Towards a Characterization of Worst Case Equilibria in the Discriminatory Price Auction

Evangelos Markakis[1] Alkmini Sgouritsa[2] Artem Tsikiridis[1]
[1]Athens University of Economics and Business, Greece
[2]University of Liverpool, UK

Abstract

We study the performance of the discriminatory price auction under the uniform bidding interface, which is one of the popular formats for running multi-unit auctions in practice. We undertake an equilibrium analysis with the goal of characterizing the inefficient mixed equilibria that may arise in such auctions. We consider bidders with capped-additive valuations, which is in line with the bidding format, and we first establish a series of properties that help us understand the sources of inefficiency under mixed strategies. Moving on, we then use these results to derive new lower and upper bounds on the Price of Anarchy of mixed equilibria. For the case of two bidders, we arrive at a complete characterization of inefficient equilibria and show an upper bound of 1.1095, which is also tight. For multiple bidders, we show that the Price of Anarchy is strictly worse, improving the best known lower bound for submodular valuations. We further present an improved upper bound of $4/3$ for the special case where there exists a "high" demand bidder. This class of instances is believed to be representative of the worst-case inefficiency, and therefore our results strengthen the perception that such auctions can work well in practice in terms of the generated social welfare. Finally, we also study Bayes-Nash equilibria, and exhibit a separation result that had been elusive so far. Namely, already with two bidders, the Price of Anarchy for Bayes-Nash equilibria is strictly worse than that for mixed equilibria. Such separation results are not always true (e.g., the opposite is known for simultaneous second price auctions) and reveal that the Bayesian model here introduces further, albeit to a small extent, inefficiency.

1 Introduction

Multi-unit auctions form a quite popular transaction mean for selling multiple identical units of a single good. They have been in use for a long time, and there are by now several practical implementations across many countries. Some of the most prominent applications involve government sales of treasury securities to investors [6], as well as electricity auctions (for distributing electrical energy) [18]. Apart from governmental use, they are also run in other financial markets, and they are being deployed in various platforms, including several online brokers [16]. In the economics literature, multi-unit auctions have been a subject of study ever since the seminal work of Vickrey [23], and some formats were conceived even earlier, by Friedman [11].

The focus of our work is on the welfare performance of the *discriminatory price auction*, which is also referred to as pay-your-bid auction. In particular, we study the *uniform bidding interface*, which is the format most often employed in practice. Under this format, each bidder submits two parameters, a monetary per-unit bid, for her willingness to pay per unit, along with an upper bound on the number of units desired. Hence, each bidder is essentially asked to declare a *capped-additive* curve (a special case of submodular functions). Given the bids, the auctioneer then allocates the units by satisfying first the demand of the bidder with the highest monetary bid, then moving to the second highest bid, and so on, until there are no units left. As a price, each winning bidder pays his bid multiplied by the number of units received.

Multi-unit auctions have received considerable attention in the literature, given their practical appeal. Since these mechanisms are not truthful, in the more recent years, a few works have already studied the social welfare guarantees that can be obtained at equilibrium. The outcome of these works is quite encouraging for the discriminatory price auction. Namely, pure Nash equilibria are always efficient, whereas for mixed and Bayes-Nash equilibria, the Price of Anarchy is bounded by 1.58 [9] for submodular valuations. These results suggest that simple auction formats can attain desirable guarantees and provide theoretical grounds for the overall success in practice.

Despite however these positive findings, there has been no progress on improving the current Price of Anarchy bounds. The known lower bound of 1.109 by [8] is quite far from the upper bounds derived by the commonly used smoothness-based approaches, [9, 22], which however do not seem applicable for producing further improvements. We believe the main difficulty in getting tighter results is that one needs to delve more deeply into the properties of Nash equilibria. But obtaining any form of characterization results for mixed or Bayesian equilibria is a notoriously hard problem. Even with two bidders it is often difficult to describe how the set of equilibria look like.

1.1 Contribution

Motivated by the previous discussion, in Section 3.3 we initiate an equilibrium analysis for enhancing our understanding of mixed equilibria. We consider bidders with capped-additive valuations, which is a subclass of submodular valuations, and consistent with the bidding format. Our results can be seen as a partial characterization of inefficient mixed equilibria, and our major highlights include both structural properties on the demand profile (see Theorem 14), as well as properties on the distributions of the mixed strategies (see Corollary 20, Theorem 22 and Lemma 24).

Moving on, in Section 4, we use these results to derive new lower and upper bounds on the Price of Anarchy of mixed equilibria. For the case of two bidders, we arrive at a complete characterization of inefficient equilibria and show an upper bound of 1.1095, which is also tight. For multiple bidders, we show that the Price of Anarchy is strictly worse, which also improves the best known lower bound for submodular valuations [8]. We further present an improved upper bound of $4/3$ for the special case where there exists a "high" demand bidder. We believe these latter instances are representative of the worst-case inefficiency that may arise, and refer to the relevant discussion in Section 4.2. To summarize, our results show that in several cases, the Price of Anarchy is even lower than the previous bound of [9] and strengthen the perception that such auctions can work well in practice in terms of generated welfare.

Finally, in Section 5, we also study Bayes-Nash equilibria, and we exhibit a separation result that had been elusive so far, even under more general valuations: already with two bidders, the Price of Anarchy for Bayes-Nash equilibria is strictly worse than for mixed equilibria. Such separation results, though intuitive, do not hold for all auction formats (see e.g. simultaneous second price auctions when bidders have submodular valuations [7]) and reveal that the Bayesian model here introduces a further source of inefficiency.

1.2 Related Work

For an exposition on multi-unit auctions and their earlier applications, we refer to the books [14] and [15]. For more recent works on applications, we refer to [6, 12, 18], for treasury bonds, carbon licence auctions, and electricity auctions, respectively.

Regarding the inefficiency of equilibria, [1] was among the first works that studied the sources of inefficiency in multi-unit auctions. For the discriminatory price auction, the Price of Anarchy was later studied in [22], and the currently best upper bound has been obtained by [9], which is $e/(e-1) \approx 1.58$ for bidders with submodular valuations (both for mixed and for Bayes-Nash equilibria). These results exploit the smoothness-based techniques, developed by [19, 22]. One can also obtain slightly worse upper bounds for subadditive valuations, by using a different methodology, based on [10]. As for lower bounds, the only construction known for submodular valuations is by [8], showing that the Price of Anarchy is at least 1.109. In parallel to these results, there has been a series of works on the inefficiency of many other auction formats, ranging from multi-unit to combinatorial auctions, see among others, [4, 5, 7, 10].

Apart from social welfare guarantees, several other aspects or properties of equilibrium behavior have been studied. Recently in [17] a characterization of equilibria is given for a model where the supply of units can be drawn from a distribution. In the past, several works have focused on revenue equivalence results between the discriminatory price and the uniform price auction, see e.g. [2, 20]. On a different direction, comparisons from the perspective of the bidders is carried out in [3].

2 Notation and Definitions

We consider a discriminatory price multi-unit auction, involving the allocation of k identical units of a single item, to a set $\mathcal{N} = \{1, \dots, n\}$ of bidders. Each bidder $i \in \mathcal{N}$ has a private value $v_i > 0$ which reflects her value per unit and a private demand $d_i \in \mathbb{Z}_+$ which reflects the maximum number of units bidder i requires. Therefore, if the auction allocates $x_i \leq k$ units to bidder i, her total value will be $\min\{x_i, d_i\} \cdot v_i$. We note that this class of valuations is a subclass of submodular valuations, and includes all additive vectors (when $d_i = k$). We will refer to them as *capped-additive* valuations.

We focus on the following simple format for the discriminatory price auction, which is known as the *uniform bidding* interface. The auctioneer asks each bidder $i \in \mathcal{N}$ to submit a tuple (b_i, q_i), where $b_i \geq 0$, is her monetary bid per unit (not necessarily equal to v_i), and q_i is her demand bid (not necessarily equal to d_i). We denote by $\mathbf{b} = (b_1, \dots, b_n)$ the monetary bidding vector, and similarly $\mathbf{q}$ will be the declared demand vector. For a bidding profile $(\mathbf{b}, \mathbf{q})$, the auctioneer allocates the units by satisfying first the demand of the bidder with the highest monetary bid, then moving to the second highest bid, and so on, until there are no units left. Hence, all the winners have their demand satisfied, except possibly for the one selected last, who may be partially satisfied. Moreover, we assume that in case of ties, a deterministic tie-breaking rule is used, which does not depend on the input bids submitted by the players to the auctioneer (e.g., a fixed ordering of the players suffices).

For every bidding profile $(\mathbf{b}, \mathbf{q})$, we let $x_i(\mathbf{b}, \mathbf{q}) \leq q_i$ be the number of units allocated to bidder i. In the discriminatory auction, the auctioneer requires each bidder i to pay b_i per allocated unit, hence a total payment of $b_i \cdot x_i(\mathbf{b}, \mathbf{q})$. The utility function of bidder $i \in \mathcal{N}$ given a bidding profile $(\mathbf{b}, \mathbf{q})$ is: $u_i(\mathbf{b}, \mathbf{q}) = \min\{x_i(\mathbf{b}, \mathbf{q}), d_i\} v_i - x_i(\mathbf{b}, \mathbf{q}) b_i$.

Viewed as games, these auctions have an infinite pure strategy space, and we also allow bidders to play mixed strategies, which are probability distributions over their set of pure strategies. When each bidder $i \in \mathcal{N}$ uses a mixed strategy G_i, she *independently* draws a bid (b_i, q_i) from G_i. We refer to $\mathbf{G} = \times_{i=1}^{n} G_i$ as the product distribution of bids. Under mixed strategies, the expected utility of a bidder i is $\mathbb{E}_{(\mathbf{b},\mathbf{q})\sim\mathbf{G}}[u_i(\mathbf{b}, \mathbf{q})]$.

Definition 1. We say that $\mathbf{G}$ is a mixed Nash equilibrium when for all $i \in \mathcal{N}$, all $b'_i \geq 0$ and all $q'_i \in \mathbb{Z}_+$

$$\mathop{\mathbb{E}}_{(\mathbf{b},\mathbf{q})\sim\mathbf{G}}[u_i(\mathbf{b}, \mathbf{q})] \geq \mathop{\mathbb{E}}_{(\mathbf{b}_{-i},\mathbf{q}_{-i})\sim\mathbf{G}_{-i}}\left[u_i((b'_i, \mathbf{b}_{-i}), (q'_i, \mathbf{q}_{-i}))\right].$$

We note that in any equilibrium, if a bidder i declares with positive probability a bid that exceeds v_i, she should not be allocated any unit, since such strategies are strictly dominated by bidding the actual value v_i.

Fact 2. Let $\mathbf{G}$ be a mixed Nash equilibrium. The probability that a bidder i is allocated some units, conditioned that she bids higher than v_i, is 0.

In the sequel, we focus on equilibria, where the monetary bids never exceed the value per unit.

Given a valuation profile $(\mathbf{v}, \mathbf{d})$, we denote by $OPT(\mathbf{v}, \mathbf{d})$ the optimal social welfare (which can be computed very easily by running the allocation algorithm of the auction with the true value and demand vector). We also denote by $SW(\mathbf{G})$ the expected social welfare of a mixed Nash equilibrium $\mathbf{G}$, i.e., equal to $\mathbb{E}_{(\mathbf{b},\mathbf{q})\sim\mathbf{G}}[\sum_i \min\{x_i(\mathbf{b}, \mathbf{q}), d_i\}v_i]$. The Price of Anarchy is the worst-case ratio $\frac{OPT(\mathbf{v},\mathbf{d})}{SW(\mathbf{G})}$ over all valuation profiles $(\mathbf{v}, \mathbf{d})$, and equilibria $\mathbf{G}$.

3 Towards a Characterization of Inefficient Mixed Equilibria

In this section, we derive a series of important properties, that help us understand better how can inefficient equilibria arise. These properties will help us analyze the Price of Anarchy in Section 4. All missing proofs from this section and all subsequent sections, can be found in the Appendix.

3.1 Mixed Nash Equilibria with Demand Revelation

Our first result is that it suffices to focus on equilibria where bidders truthfully reveal their demand, resulting therefore in a single-parameter strategy space for the bidders.

Theorem 3. Let $(\mathbf{v}, \mathbf{d})$ be a valuation profile, and $\mathbf{G}$ be a mixed Nash equilibrium. Then, for every $i \in \mathcal{N}$, and in every pure strategy profile $(b_i, q_i) \sim G_i$, we can replace q_i by d_i so that the resulting distribution remains a mixed Nash equilibrium with the same social welfare.

We prove the above theorem by a series of lemmas. The first step is the next lemma, showing that it suffices to consider only equilibria, where nobody declares a demand bid that is lower than their true demand.

Lemma 4. Let $\mathbf{G}$ be any mixed Nash equilibrium and G'_i be the same as G_i after replacing any $q_i < d_i$ with d_i. Then $\mathbf{G}'$ is also a mixed Nash equilibrium with the same social welfare.

The next step is to prove (in Lemma 5) that if $\sum_i d_i > k$, then it is sufficient to consider only Nash equilibria, where nobody declares more demand that their true demand.

Lemma 5. Suppose that $\sum_i d_i > k$ and let $\mathbf{G}$ be any mixed Nash equilibrium where nobody declares less demand and G'_i be the same as G_i after replacing any $q_i > d_i$ with d_i. Then $\mathbf{G}'$ is also a mixed Nash equilibrium with the same social welfare.

The remaining case that has not been covered by Lemma 5, is when the total demand does not exceed k: $\sum_i d_i \leq k$. But as we show below, these are efficient equilibria.

Lemma 6. If $\sum_i d_i \leq k$ then the social welfare of any mixed Nash equilibrium is optimal.

Proof of Theorem 3. The proof follows by combining Lemmas 4, 5 and 6. □

3.2 Existence of Non-empty-handed Bidders

For the rest of the paper we consider only strategy profiles where the bidders' demand bid matches their true demand. The main goal of this subsection is to derive Theorem 14, which is a crucial property for understanding the formation of inefficient mixed equilibria. To proceed, we give first some further notation to be used in this and the following sections.

Further notation. Given Theorem 3, instead of using distributions on tuples (b_i, q_i), we suppose that each bidder $i \in \mathcal{N}$ independently draws only a monetary bid b_i from a distribution B_i and we refer to $\mathbf{B} = \times_{i=1}^{n} B_i$ as the product distribution of monetary bids or just bids from now on. For a bidding profile $\mathbf{b}$, the utility of a bidder i will simply be denoted as $u_i(\mathbf{b})$, instead of $u_i(\mathbf{b}, \mathbf{d})$. Definition 1 is also simplified, and we say that $\mathbf{B}$ is an equilibrium if $\mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})] \geq \mathbb{E}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i((b'_i, \mathbf{b}_{-i}))]$, for any i and any $b'_i \geq 0$. Similarly, the social welfare of a mixed Nash equilibrium $\mathbf{B}$ is given by just $SW(\mathbf{B})$ instead of $SW(\mathbf{G})$.

For a mixed strategy bidding profile $\mathbf{B}$, we denote by $W(\mathbf{B})$ the set of bidders with positive expected utility, i.e., $W(\mathbf{B}) = \{j : \mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_j(\mathbf{b})] > 0\}$, and let $\mathbf{B}_W = \times_{i\in W(\mathbf{B})} B_i$. Moreover, the support of a bidder i in $\mathbf{B}$ is the domain of the distribution B_i, that i plays under $\mathbf{B}$, denoted by $Supp(B_i)$. We denote by $\ell(B_i), h(B_i)$ the leftmost and rightmost points in the support of bidder i. In particular, if the rightmost part of the domain of B_i is a mass point b or an interval $[a, b]$ then $h(B_i) = b$, and similarly for $\ell(B_i)$ (in cases of distributions over intervals, we can always assume that the domain contains only *closed* intervals, because the endpoints are chosen with zero probability). We further denote by $\ell(\mathbf{B}_W), h(\mathbf{B}_W)$ the leftmost and rightmost points of the union of all the supports of $W(\mathbf{B})$.

For $i = 1, \ldots, n$ we denote by F_i the CDF of B_i and by f_i their PDF. Moreover, given a profile $\mathbf{b}$, it is often useful in the analysis to think of the vector of winning bids that a bidder i faces, denoted by $\beta(\mathbf{b})_{-i} = (\beta_1(\mathbf{b}_{-i}), \ldots, \beta_k(\mathbf{b}_{-i}))$. Here $\beta_j(\mathbf{b}_{-i})$ is the j-th lowest winning bid of the profile $\mathbf{b}_{-i}$, for $j = 1, \ldots, k$. This implies that, under profile $\mathbf{b}$, bidder i is allocated $j = 1, \ldots, k-1$ units when $\beta_j(\mathbf{b}_{-i}) < b_i < \beta_{j+1}(\mathbf{b}_{-i})$ and k units when $\beta_k(\mathbf{b}_{-i}) < b_i$. We note that because we focus on the uniform bidding interface, some consecutive β_j values may coincide and be equal to the bid of the same winning bidder. When $\mathbf{b}_{-i} \sim \mathbf{B}_{-i}$, for $i = 1, \ldots, n$ and $j = 1, \ldots, k$, we denote the CDF of the random variable $\beta_j(\mathbf{b}_{-i})$ as $\hat{F}_{ij}$.

Fact 7. Let $\mathbf{B}_{-i}$ be a product distribution of bids. Then for all $\alpha \geq 0$, where no bidder other than (possibly) i has a mass point,

$$\mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[x_i(\alpha, \mathbf{b}_{-i})] = \sum_{j=1}^{d_i} \hat{F}_{ij}(\alpha)\,.$$

Given a bidding profile $\mathbf{B}$, for any bidder i we define

$$\hat{F}_i^{avg}(x) = \frac{\sum_{j=1}^{d_i} \hat{F}_{ij}(x)}{d_i}\,,$$

to be the average CDF of the winning bids that bidder i competes against. Note that $\hat{F}_i^{avg}$ is a CDF since it is the average of a number of CDFs.

Remark 8. We remark that the $\hat{F}_{ij}$ functions are right continuous as CDFs and moreover if the F_i functions have no mass point, the same holds for the $\hat{F}_{ij}$ functions. Additionally, if for any i the $\hat{F}_{ij}$ functions are continuous, so the $\hat{F}_i^{avg}$ is as the average of continuous functions.

We start by ruling out certain scenarios that cannot occur at inefficient equilibria. Our first finding is that we can safely ignore bidders with zero expected utility, since in any *inefficient* mixed Nash equilibrium they do not receive any units.

Lemma 9. Any mixed Nash equilibrium $\mathbf{B}$ with at least one bidder with zero expected utility, but positive expected number of allocated units, is efficient.

Next, we show that to have inefficiency at an equilibrium, there must exist at least two bidders with positive expected utility.

Lemma 10. Let $(\mathbf{v}, \mathbf{d})$ be a valuation profile and $\mathbf{B}$ be an inefficient mixed Nash equilibrium. Then, $|W(\mathbf{B})| \geq 2$.

The next warm-up properties involve the expected utility of a bidder under an equilibrium $\mathbf{B}$, conditioned that she bids within a certain interval or at a single point. We start with Fact 11, which is a straightforward implication of the equilibrium definition, and proceed by arguing that no two bidders may bid on the same point with positive probability. Theorem 13 concludes by stating the main property regarding the utility of bidders when bidding in their support.

Fact 11. Let $\mathbf{B}$ be an equilibrium. For a bidder i, consider a partition of $Supp(B_i)$ (or of a subset of it) into smaller disjoint sub-intervals, say $I_1, \ldots, I_\ell$, such that B_i has a positive probability on each sub-interval (mass points may also be considered as sub-intervals with zero measure). Then, it should hold that $\mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b}) \mid b_i \in I_r] = \mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})]$ for every $r = 1, \ldots, \ell$.

Observation 12. In any mixed Nash equilibrium $\mathbf{B}$ there can be no bidders $i, j \in W(\mathbf{B})$ and point z such that $Pr[b_i = z] > 0$ and $Pr[b_j = z] > 0$.

Based on Fact 11, we can obtain the following point-wise version. Variations of the version below have also appeared in related works, see e.g., [8].

Theorem 13. Consider a bidder i, and a mixed Nash equilibrium $\mathbf{B}$. Then for any $z \in Supp(B_i)$, where no other bidder has a mass point, $\mathbb{E}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(z, \mathbf{b}_{-i})] = \mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})]$.

The main theorem of this section follows, stating the existence of a special bidder that always receives at least one unit that we call *non-empty-handed.*

Theorem 14. Let $(\mathbf{v}, \mathbf{d})$ be a valuation profile, and let $\mathbf{B}$ be a mixed Nash Equilibrium. If $W(\mathbf{B}) \geq 1$, then there exists a bidder $i \in W(\mathbf{B})$, such that

$$\sum_{j \in W(\mathbf{B}) \setminus \{i\}} d_j \leq k - 1 \, .$$

Proof. On the contrary, assume that for every $i \in W(\mathbf{B})$, it holds that $\sum_{j \in W(\mathbf{B}) \setminus \{i\}} d_j \geq k$. Let i be a bidder for which $\ell = \ell(\mathbf{B}_W) \in Supp(B_i)$. We will distinguish the following cases.
Case 1: There exists an interval in the form $[\ell, \ell + \epsilon]$, on which B_i has a positive probability mass and on which the bidders of $W(\mathbf{B}) \setminus \{i\}$ have a zero mass. We note that we also allow $\epsilon = 0$, i.e., that i has a mass point on ℓ and the other bidders do not. This means that when bidder i bids within $[\ell, \ell + \epsilon]$, all the other bidders from $W(\mathbf{B})$ are above him. Since we assumed that the total demand of $W(\mathbf{B}) \setminus \{i\}$ is at least k, this means that bidder i does not win any units in this case. Since i bids with positive probability in $[\ell, \ell + \epsilon]$, $\mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})] = 0$, which is a contradiction to Fact 11.
Case 2: By Observation 12 it cannot be the case that bidder i and some other bidder have a mass point on ℓ.
Case 3: Any mass point that may exist by the bidders is at a point $x > \ell$, and there is also no interval starting from ℓ that is used only by bidder i. Hence, there exists an interval I in the form $I = [\ell, \ell + \epsilon]$ for some small enough $\epsilon > 0$, and a bidder $j \in W(\mathbf{B}) \setminus \{i\}$, such that both B_i and B_j contain I in their support, and have positive probability mass on I without mass points (there can be even more than just one such bidder j).

By Theorem 13, we obtain that $\mathbb{E}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(\ell, \mathbf{b}_{-i})] = u$. But this is a contradiction, because by bidding ℓ, bidder i ranks lower than all other bidders of $W(\mathbf{B})$ with probability one. By our assumption that $\sum_{j \in W(\mathbf{B}) \setminus \{i\}} d_j \geq k$, there are no units left for i when she ranks last among $W(\mathbf{B})$, and therefore, $\mathbb{E}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(\ell, \mathbf{b}_{-i})] = 0 \neq u$. ☐

Next we give as a corollary that if all bidders have unit demand, any mixed Nash equilibria is efficient.

Corollary 15. Let $(\mathbf{v}, \mathbf{d})$ be a valuation profile with only unit-demand bidders, i.e., $d_i = 1$ for all i. Then any mixed Nash equilibrium $\mathbf{B}$ is efficient.

Proof. If $|W(\mathbf{B})| < 2$ then, by Lemma 10, $\mathbf{B}$ is efficient. If $|W(\mathbf{B})| \geq 2$, by Theorem 14 there exist at most k bidders with positive expected utility. The rest of the bidders have zero expected allocation by Lemma 9, otherwise $\mathbf{B}$ is efficient. The only way that $\mathbf{B}$ is inefficient is if there exist bidders i and j with $v_i < v_j$ and $i \in W(\mathbf{B})$ whereas $j \notin W(\mathbf{B})$. In such a case, $\mathbb{E}[x_i(\mathbf{b})] > 0$ and if bidder j bids $v_i + \varepsilon < v_j$, $\mathbb{E}[x_j(v_i + \varepsilon, \mathbf{b}_{-j})] > \mathbb{E}[x_i(\mathbf{b})] > 0$, which results in a positive expected utility for bidder j contradicting the fact that $\mathbf{B}$ is a Nash equilibrium. □

3.3 Properties of the support and the CDFs of Mixed Nash Equilibria

Theorem 14, that guarantees the existence of a non-empty-handed bidder, will help us to establish a set of further properties that characterize the structure of inefficient mixed Nash equilibria. These properties (and especially Theorem 22) will be important to establish the inefficiency results that follow.

We start with an observation regarding the highest bid of any bidder $i \in W(\mathbf{B})$ which should be strictly less than v_i.

Observation 16. For any bidder $i \in W(\mathbf{B})$, $h(B_i) < v_i$.

Proof. Suppose on the contrary that for some $i \in W(\mathbf{B})$, $h(B_i) = v_i$. Then, by Theorem 13, $\mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})] = \mathbb{E}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(v_i, \mathbf{b}_{-i})] = 0$, which contradicts the fact that $i \in W(\mathbf{B})$. □

The next lemma shows that at any equilibrium $\mathbf{B}$, bidders who are not non-empty-handed cannot have higher bids in their support than the support of the non-empty-handed bidders. Additionally, any bidder who is non-empty-handed does not have a reason to use bids that are higher than the maximum bids of all other winning bidders. The reason is that if such differences existed, then there would be incentives to win the same number of units by lowering one's bid.

Lemma 17. Let $(\mathbf{v}, \mathbf{d})$ be a valuation profile and $\mathbf{B}$ be an inefficient mixed Nash equilibrium. Then, for any non-empty-handed bidder i, it holds that $h(B_i) = h(\mathbf{B}_{W\setminus\{i\}}) = h(\mathbf{B}_W)$.

Next we show that no bidder may be the only bidder bidding in any point or interval.

Lemma 18. Let $(\mathbf{v}, \mathbf{d})$ be a valuation profile and $\mathbf{B}$ be a mixed Nash equilibrium. For all $i \in W(\mathbf{B})$, it holds that $Supp(B_i) \subseteq \bigcup_{j\in W(\mathbf{B})\setminus\{i\}} Supp(B_j)$.

Lemma 19. Let $(\mathbf{v}, \mathbf{d})$ be a valuation profile and $\mathbf{B}$ be an inefficient mixed Nash equilibrium.
1) There exists no bidder $i \in W(\mathbf{B})$ and no point $z \in Supp(B_i) \setminus \{\ell(\mathbf{B}_W)\}$, with $F_i(z) > \lim_{z\to z^-} F_i(z)$, i.e., there are no mass points among the bidders of $W(\mathbf{B})$, except possibly the leftmost endpoint of all bidders' distributions.
2) At most one bidder $i \in W(\mathbf{B})$ may have a mass point on $\ell(\mathbf{B}_W)$, i.e., $Pr[b_i = \ell(\mathbf{B}_W)] > 0$, and i is a non-empty-handed.

By combining Theorem 13 and Lemma 19 we get the following Corollary.

Corollary 20. For any inefficient mixed Nash equilibrium $\mathbf{B}$ the following hold:
1) For any bidder i and $z \in Supp(B_i) \setminus \{\ell(\mathbf{B}_W)\}$, $\mathbb{E}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(z, \mathbf{b}_{-i})] = \mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})]$.
2) If there exists a bidder i with $Pr[b_i = \ell(\mathbf{B}_W)] > 0$, then i is a non-empty-handed bidder and $\mathbb{E}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(\ell(\mathbf{B}_W), \mathbf{b}_{-i})] = \mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})]$.
3) If no non-empty-handed bidder exists with mass point on $\ell(\mathbf{B}_W)$, for any bidder i with $\ell(\mathbf{B}_W) \in Supp(B_i)$, $\mathbb{E}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(\ell(\mathbf{B}_W), \mathbf{b}_{-i})] = \mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})]$.

Observation 21. For any inefficient mixed Nash equilibrium $\mathbf{B}$, either there exists a bidder $i \in W(\mathbf{B})$ with mass point on $\ell(\mathbf{B}_W)$ and this is a non-empty-handed bidder, or there are at least two non-empty-handed bidders of $W(\mathbf{B})$ with $\ell(\mathbf{B}_W)$ in their support.

Proof. If there exists a bidder i with mass point on $\ell(\mathbf{B}_W)$, then by Lemma 19 i is a non-empty-handed bidder. If there is no such bidder then we argue that no bidder $j \in W(\mathbf{B})$ that is not non-empty-handed has $\ell(\mathbf{B}_W)$ in their support.

Suppose on the contrary that $\ell(\mathbf{B}_W) \in Supp(B_j)$ for some bidder j that is not non-empty-handed. Then since no bidder has a mass point on $\ell(\mathbf{B}_W)$, everybody bids above $\ell(\mathbf{B}_W)$ with probability one, leaving j with no units while bidding $\ell(\mathbf{B}_W)$. By Corollary 20, $\mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_j(\mathbf{b})] = \mathbb{E}_{\mathbf{b}_{-j}\sim\mathbf{B}_{-j}}[u_j(\ell(\mathbf{B}_W), \mathbf{b}_{-j})] = 0$ contradicting the fact that $j \in W(\mathbf{B})$.

By Lemma 18 there are at least two bidders bidding on $\ell(\mathbf{B}_W)$, which concludes the proof. $\square$

Given any (inefficient) equilibrium, the next theorem specifies the average CDF of the winning bids that bidder i competes against, i.e., $\hat{F}_i^{avg}$, in i's support.

Theorem 22. Let $(\mathbf{v}, \mathbf{d})$ be a valuation profile and $\mathbf{B}$ be an inefficient mixed Nash equilibrium. Then, for $i \in W(\mathbf{B})$, the CDF $\hat{F}_i^{avg}$ satisfies

$$\hat{F}_i^{avg}(z) = \frac{u_i}{d_i(v_i - z)}\,, \quad \forall z \in Supp(B_i)\,,$$

for some constant $u_i = \mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})] > 0$.

Proof. Fix a bidder $i \in W(\mathbf{B})$. For all intervals $I \subseteq Supp(B_i)$, by Corollary 20 it must be that for all $z \in I \setminus \ell(\mathbf{B}_W)$

$$\mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(z, \mathbf{b}_{-i})] = \mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})] = u_i > 0\,.$$

The above equality is equivalent to

$$\mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[x_i(z, \mathbf{b}_{-i})] = \frac{u_i}{v_i - z} \Leftrightarrow \hat{F}_i^{avg}(z) = \frac{u_i}{d_i(v_i - z)}\,,$$

for all $z \in Supp(B_i)$. The last equivalence is due to Fact 7. The theorem follows since $\hat{F}_i^{avg}$ is right continuous. $\square$

A corollary of Theorem 22 is that the union of the support of the winners is an interval.

Corollary 23. Let $(\mathbf{v}, \mathbf{d})$ be a valuation profile and $\mathbf{B}$ be an inefficient mixed Nash equilibrium. Then, for every bidder $i \in W(\mathbf{B})$, $\bigcup_{j\in W(\mathbf{B})\setminus\{i\}} Supp(B_j) = [\ell(\mathbf{B}_W), h(\mathbf{B}_W)]$.

The final lemma of this section shows that the rightmost point in the support of $\mathbf{B}$ can be expressed as a function of the parameters of certain non-empty-handed bidders.

Lemma 24. Let $(\mathbf{v}, \mathbf{d})$ be a valuation profile and $\mathbf{B}$ be an inefficient mixed Nash equilibrium. Let $i \in W(\mathbf{B})$ be the non-empty-handed bidder such that $Pr[b_i = \ell(\mathbf{B}_W)] > 0$, or if no such bidder exists, then let i be any non-empty-handed bidder with $\ell(\mathbf{B}_W)$ in his support. We have

$$h(\mathbf{B}_W) = h(B_i) = v_i - (k - \sum_{j\in W(\mathbf{B})\setminus\{i\}} d_j)\frac{v_i - \ell(\mathbf{B}_W)}{d_i}\,.$$

Proof. Let i be the bidder specified by the lemma's statement; note that such a bidder always exists by Observation 21. By Lemma 17, $h(\mathbf{B}_W) = h(B_i)$, and by applying Corollary 20, it must be that

$$\mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(\ell(B_i), \mathbf{b}_{-i})] = \mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(h(B_i), \mathbf{b}_{-i})]\,. \tag{1}$$

Moreover,

$$\mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(h(B_i), \mathbf{b}_{-i})] = \mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[x_i(h(B_i), \mathbf{b}_{-i})](v_i - h(B_i)) = d_i(v_i - h(B_i))\,. \tag{2}$$

Equation (2) holds since bidding $h(B_i)$ guarantees outbidding every other bidder in the auction and thus grants d_i units to i (recall that there is no mass point on $h(B_i)$ due to Lemma 19, and therefore the event of losing due to tie-breaking by bidding $h(B_i)$ has probability zero).

On the other hand, note that by the way i has been defined, $\ell(B_i) = \ell(\mathbf{B}_W)$ and therefore

$$\begin{aligned}
\mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim B_{-i}}[u_i(\ell(B_i), \mathbf{b}_{-i})] &= \mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim B_{-i}}[u_i(\ell(\mathbf{B}_W), \mathbf{b}_{-i})] \\
&= \mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim B_{-i}}[x_i(\ell(\mathbf{B}_W), \mathbf{b}_{-i})](v_i - \ell(\mathbf{B}_W)) \\
&= (k - \sum_{j\in W(\mathbf{B})\setminus\{i\}} d_j)(v_i - \ell(\mathbf{B}_W))\,,
\end{aligned} \tag{3}$$

where in the last equality above, we have that $k - \sum_{j\in W(\mathbf{B})\setminus\{i\}} d_j > 0$, since i is non-empty-handed. By equating now (2) and (3), the lemma follows. $\square$

4 Price of Anarchy for mixed equilibria

In this section, we exploit the properties derived so far for mixed Nash equilibria, in order to analyze the inefficiency of the discriminatory price auction. Since we will be dealing with inefficient equilibria, we assume that in any valuation profile considered in this section, there are at least two bidders with a different value per unit.

4.1 The case of two bidders

We pay particular attention to the case of $n = 2$. This is a setting where we can fully characterize in closed form the distributions of the inefficient mixed Nash equilibria, and derive valuable intuitions for the worst-case instances with respect to the Price of Anarchy, that are helpful also for auctions with multiple bidders. The main result of this subsection is the following theorem, showing that the inefficiency is quite limited.

Theorem 25. For $k \geq 2$, $n = 2$ and capped additive valuation profiles, the Price of Anarchy of mixed equilibria is at most 1.1095, and this is tight as k goes to infinity.

We postpone the proof of Theorem 25, as we first need to establish some properties regarding the form of inefficient mixed Nash equilibria with two bidders. For $n = 2$, a capped-additive valuation profile can be described as $(\mathbf{v}, \mathbf{d}) = ((v_1, d_1), (v_2, d_2))$. Recall also that it is sufficient to focus our attention only on profiles where $d_1 + d_2 > k$, since otherwise, by Lemma 6 any mixed equilibrium is efficient.

We start our analysis by characterizing the support of inefficient mixed Nash equilibria.

Lemma 26. Let $(\mathbf{v}, \mathbf{d}) = ((v_1, d_1), (v_2, d_2))$ be any capped-additive valuation profile of two bidders and $\mathbf{B} = (B_1, B_2)$ be any inefficient mixed Nash equilibrium. Then:

1. $Supp(B_1) = Supp(B_2) = [\ell(B_1), h(B_1)]$, and $\ell(B_1) = 0$.

2. $h(B_1)$ takes one of the following values

$$h(B_1) = v_1 \frac{d_1 + d_2 - k}{d_1} \quad \text{or} \quad h(B_1) = v_2 \frac{d_1 + d_2 - k}{d_2} \, .$$

The following theorem specifies the cumulative distribution functions that comprise any inefficient mixed Nash equilibrium, along with a necessary condition for the existence of such equilibria. For a bidder i below, we use the notation v_{-i} and d_{-i} to denote the value and demand of the other bidder.

Theorem 27. Let $(\mathbf{v}, \mathbf{d}) = ((v_1, d_1), (v_2, d_2))$ be a capped-additive valuation profile of two bidders and $\mathbf{B} = (B_1, B_2)$ any inefficient mixed Nash equilibrium.

1. The cumulative distribution function of bidder i, for $i = 1, 2$, is

$$F_i(z) = \frac{1}{d_1 + d_2 - k} \left(\frac{d_{-i}(v_{-i} - h(B_i))}{v_{-i} - z} - (k - d_i) \right) . \tag{4}$$

2. Furthermore, for i being the non-empty-handed bidder with a mass point at 0, or if no such bidder exists, being any non-empty-handed bidder, it holds that $\frac{v_{-i}}{v_i} \geq \frac{d_{-i}}{d_i}$,

Proof. It is convenient to look into the $\hat{F}_i^{avg}$ distribution to derive the claimed formulas. In the two-bidder environment, the sole source of competition is a single bidder. Firstly, for any bidder i, the other bidder always obtains $k - d_i$ units[1] and there is a competition for the remaining $d_i - (k - d_{-i}) = d_1 + d_2 - k$ units. Therefore, for the competitor of bidder i, we have:

$$\hat{F}_{-i}^{avg}(z) = \frac{\sum_{j=1}^{d_{-i}} \hat{F}_{-i,j}(z)}{d_{-i}} = \frac{k - d_i + (d_1 + d_2 - k) F_i(z)}{d_{-i}} \, .$$

Now by Theorem 22, for $z \in Supp(B_{-i})$ (which is the same as $Supp(B_i)$ by Lemma 26), we have

$$\hat{F}_{-i}^{avg}(z) = \frac{u_{-i}}{d_{-i}(v_{-i} - z)} \, ,$$

for some constant $u_{-i} > 0$. By combining the last two equations and rearranging terms we obtain

$$F_i(z) = \frac{1}{d_1 + d_2 - k} \left(\frac{u_{-i}}{v_{-i} - z} - (k - d_i) \right) \, .$$

We now determine the appropriate value for $u_{-i} > 0$ so that $F_i(z)$ is a valid cumulative distribution function in $Supp(B_i)$. It must be that $F_i(h(B_i)) = 1$, since $h(B_i)$ is the rightmost point in her support. Hence,

$$1 = \frac{1}{d_1 + d_2 - k} \left(\frac{u_{-i}}{v_{-i} - h_i(B_i)} - (k - d_i) \right) \Leftrightarrow u_{-i} = d_{-i}(v_{-i} - h_i(B_i)) \, .$$

This establishes that $F_i(z)$ satisfies Equation (4).

The second part of the theorem comes from the fact that the CDFs should also satisfy non-negativity. For this, let i be the bidder specified by the second part of the theorem's statement. Then, by using Lemma 24 it is a matter of simple calculations to see that $F_i(0) \geq 0$ is equivalent to $\frac{v_{-i}}{v_i} \geq \frac{d_{-i}}{d_i}$. Since F_i is increasing, we have established that this condition is necessary to enforce that $F_i(z) \geq 0$ for every $z \in Supp(B_i)$. $\square$

[1] This is consistent with Theorem 14 since when a bidder is not non-empty-handed, it must be that the demand of the other bidder is k, and hence, she obtains $k - d_i = 0$ free units, as expected.

Remark 28. By Lemma 26 and Theorem 27, we can see that there can be at most two inefficient equilibria, depending on how the interval of the support was determined.

Given an equilibrium $\mathbf{B}$, let $SW(\mathbf{B})$ be the expected social welfare derived by $\mathbf{B}$. The next lemma expresses the social welfare, as a function of the relevant parameters of a profile.

Lemma 29. Let $(\mathbf{v}, \mathbf{d}) = ((v_1, d_1), (v_2, d_2))$ be a capped-additive valuation profile and $\mathbf{B}$ be an inefficient mixed equilibrium.. Let i be a non-empty-handed bidder such that $Pr[b_i = 0] > 0$ or, if no such bidder exists, let i be any non-empty-handed bidder. Then, the expected social welfare of the inefficient mixed Nash equilibrium $\mathbf{B}$ is

$$SW(\mathbf{B}) = d_{-i}(v_{-i}-v_i)\left(1 - \int_0^{h(B_i)} \frac{1}{d_1+d_2-k}\left(\frac{d_i(v_i - h(B_i))}{v_i - z} - (k - d_{-i})\right)\frac{v_{-i} - h(B_i)}{(v_{-i} - z)^2}dz\right)+kv_i\,.$$

We are now ready to prove Theorem 25.

Proof of Theorem 25. The properties established so far imply a full characterization of instances that have inefficient equilibria. To establish Theorem 25, we will group instances into three appropriate classes and we will solve an appropriately defined optimization problem that approximates the Price of Anarchy for each subclass to arbitrary precision.

WLOG, suppose we are given a value profile $(\mathbf{v}, \mathbf{d}) = ((v_1, d_1), (v_2, d_2))$ of k units such that $d_1 \geq d_2$. We define the following two quantities, which we refer to as the *normalized* demands,

$$\bar{d_1} = \frac{d_1}{k} > 0\,, \qquad \bar{d_2} = \frac{d_2}{k} > 0\,. \tag{5}$$

Essentially, we intend to use $v_1, v_2, \bar{d_1}$ and $\bar{d_2}$ as the variables of the optimization problem mentioned before.

Let $\mathbf{B}$ be any inefficient mixed Nash equilibrium. With a slight abuse of notation we view the term $h(B_i)$ as a function of the valuation profile parameters, as established by Lemma 26, and define the functions $h_i(\mathbf{v}, \bar{\mathbf{d}}) = v_i\frac{\bar{d_1}+\bar{d_2}-1}{\bar{d_i}}$ for $i = 1, 2$. We pair these functions with two additional expressions $SW_i(\mathbf{v}, \mathbf{d})$ for $i = 1, 2$ which are (scaled) restatements of the social welfare of an equilibrium (as stated in Lemma 29), solely in terms of the value profile $(\mathbf{v}, \mathbf{d})$ and k, and without dependencies on the underlying equilibrium distributions. The reason we are able to do so, is Theorem 27, which tells us what the CDFs are, in terms of the valuation profile. The exact form of $SW_i(v_1, v_2, \bar{d_1}, \bar{d_2})$ for $i = 1, 2$ is

$$SW_i(\mathbf{v}, \bar{\mathbf{d}}) = \bar{d}_{-i}(v_{-i}-v_i)\left(1 - \int_0^{h_i(\mathbf{v},\bar{\mathbf{d}})} \frac{1}{\bar{d_1}+\bar{d_2}-1}\left(\frac{\bar{d_i}(v_i - h_i(\mathbf{v},\bar{\mathbf{d}}))}{v_i - z} - (1 - \bar{d}_{-i})\right)\frac{v_{-i} - h_i(\mathbf{v},\bar{\mathbf{d}})}{(v_{-i} - z)^2}dz\right)+v_i\,.$$

To continue, let i be a non-empty handed bidder with a mass point at 0 or if no such bidder exists, let i be any non-empty-handed bidder. Then,

$$kSW_i(\mathbf{v}, \bar{\mathbf{d}}) = SW(\mathbf{B})\,. \tag{6}$$

To conclude the proof of the upper bound we solve three optimization problems that we distinguish based on different cases for the values of the supports; we refer the reader to the appendix for the full proof.

By solving numerically those optimization problems we found out that in the worst case instance $v_1 = 1, v_2 \approx 0.526, \bar{d_1} = 1, \bar{d_2} \approx 0.357$. It is not hard to convert the variables to the underlying worst case instance, which we present in the next paragraph.

Tight Example. Consider an instance of the discriminatory auction for $k \geq 2$ units and $n = 2$ bidders. Bidder 1 has value $v_1 = 1$ and $d_1 = k$, whereas bidder 2 has a value $v_2 = 0.526$ and $d_2 = \lceil 0.357k \rceil$ units. Let B_1, B_2 be two distributions supported in $[0, \frac{d_2}{k}]$. Note that $v_2 > \frac{d_2}{k}$. In accordance to Equation (4), the cumulative distribution functions of B_1 and B_2 are

$$F_1(z) = \frac{v_2 - \frac{d_2}{k}}{v_2 - z}, \qquad\qquad F_2(z) = \frac{k - d_2}{d_2} \frac{z}{1 - z}.$$

It is easy to verify that $\mathbf{B} = (B_1, B_2)$ is indeed a mixed equilibrium. The optimal allocation is for bidder 1 to obtain all k units and the expected social welfare of $\mathbf{B}$ can be easily derived using Lemma 29. The worst case inefficiency ratio occurs as k grows and is approximately 1.1095. □

4.2 Multiple Bidders

We move now to instances with more than two bidders. Inspired by the construction in the previous section, we provide first a lower bound on the Price of Anarchy. This bound shows a separation between $n = 2$ and $n > 2$ in the sense that equilibria can be more inefficient with a higher number of bidders. It also improves the best known lower bound of the discriminatory price auction for the class of submodular valuations, which was 1.109, by [8]. However, the improvement is extremely small.

Theorem 30. For $n > 2$, and for the class of mixed strategy Nash equilibria, PoA ≥ 1.1204.

The above bound is the best lower bound we have been able to establish, even after some extensive experimentation (driven by the results in the remainder of this section). It is natural to wonder if one can have a matching upper bound, which would establish that the Price of Anarchy remains very small even for a large number of bidders. Recall that from [9], we know already a bound of $e/(e-1) \approx 1.58$. Although we have not managed to provide an upper bound that covers all instances, we will provide an improved upper bound for a special case, for which there is some evidence that it captures worst-case scenarios of inefficiency. At the same time, we will be able to characterize the format of such worst case equilibria in these instances.

To obtain some intuition, it is instructive to look at our two lower bounds, the first one in Section 4.1, and the second one in Theorem 30. One notices that the main source of inefficiency is the fact that the auctioneer accepts multi-unit demand declarations. When this does not occur (i.e. each bidder requires one unit), we have already shown in Corollary 15 that mixed Nash equilibria attain optimal welfare. When multi-demand bidders are present, Theorem 25 shows that in the case of two bidders, the most inefficient mixed Nash equilibrium occurs when a participating bidder declares a demand for all the units, whereas the opponent requires a much smaller fraction of the supply. In the example of Theorem 30 above, we have extended this paradigm for multiple bidders with an arbitrary demand structure but under the assumption that one of the bidders requires all the units (the additive bidder). Such a setting, of one large-demand bidder facing competition by multiple small-demand bidders has also been discussed in [3]. Furthermore, there exist other auction formats that have needed such a demand profile at their worst case instances, see e.g., [5] for tight examples of the uniform price auction. To summarize, it seems unlikely that the worst instances involve only bidders with low demand or bidders with small variation on their demands.

Given the above, in the remainder of this section, we will analyze the family of instances where there exists an additive bidder (with demand equal to k) and where she also has the highest value per unit (in fact the latter assumption is needed only for the Price of Anarchy analysis but not for the characterization of the worst-case demand profile and the equilibrium strategies). We strongly believe that this class is representative of the most inefficient mixed Nash equilibria (which is true already for the case of two bidders).

The main result of this section is the following.

Theorem 31. Consider the class of valuation profiles, where there exists an additive bidder α with the highest value, and an equilibrium $\mathbf{B}$, such that $\alpha \in W(\mathbf{B})$. Then, the Price of Anarchy is at most $4/3$.

The proof of the theorem is by following a series of steps. The existence of the additive bidder helps in the analysis, because a direct corollary of Theorem 14 is that the additive bidder is the sole non-empty-handed bidder (everyone else faces competition for all the units).

Corollary 32 (by Theorem 14)**.** Consider a valuation profile $(\mathbf{v}, \mathbf{d})$ with an additive bidder α, that admits an equilibrium $\mathbf{B}$, such that $\alpha \in W(\mathbf{B})$. Then, bidder α is the unique non-empty-handed bidder under $\mathbf{B}$, thus,

$$\sum_{i \in W(\mathbf{B}) \setminus \{\alpha\}} d_i \leq k - 1 \,.$$

To proceed, we ensure that for the instances described by Theorem 31, it suffices to analyze the equilibria where the bidder α belongs to $W(\mathbf{B})$, i.e., there cannot exist a more inefficient equilibrium $\mathbf{B}'$ of these instances where $\alpha \notin W(\mathbf{B}')$. This is addressed by the following lemma.

Lemma 33. Consider a valuation profile, and suppose that it admits two distinct inefficient equilibria, $\mathbf{B}$ and $\mathbf{B}'$. If $i \in W(\mathbf{B})$ is a non-empty-handed bidder in $\mathbf{B}$, then $i \in W(\mathbf{B}')$.

Proof. Let $i \in W(\mathbf{B})$ be a non-empty-handed bidder in $\mathbf{B}$ and suppose for contradiction that $i \notin W(\mathbf{B}')$. We know that $\sum_{j \in W(\mathbf{B}) \setminus \{i\}} d_j \leq k-1$. Since $\mathbf{B}'$ is inefficient, and i does not belong to $W(\mathbf{B}')$, by Lemma 6, there must exist a bidder m such that $m \in W(\mathbf{B}') \setminus W(\mathbf{B})$.

We can now look more closely on the bidding behavior of bidders i and m in $\mathbf{B}'$. Since $i \notin W(\mathbf{B}')$, by Lemma 9 we know that i ranks lower than all other winning bidders with probability one. From this, we claim that $Pr_{b_m \sim B'_m}[b_m \geq v_i] > 0$. Indeed if this was not the case, then $Pr_{b_m \sim B'_m}[b_m < v_i] = 1$, and bidder i would have an incentive to outbid bidder m by bidding a value lower than v_i and obtain positive utility, which is a contradiction. This implies that $h(B'_m) \geq v_i$. By Observation 16 on the maximum bid submitted by the players of $W(\mathbf{B}')$, this yields that $v_m > v_i$.

To obtain a contradiction, we come back to the equilibrium $\mathbf{B}$. Again by Observation 16, $h(B_i) < v_i$, and therefore, $Pr_{b_i \sim B_i}[b_i < v_m] = 1$. But this implies that bidder m has an incentive to outbid bidder i and obtain a positive utility, which completes the proof. $\square$

Lemma 33 and Corollary 32 guarantee that to prove an upper bound for the instances described by Theorem 31, we can focus only on the equilibria where the additive bidder belongs to $W(\mathbf{B})$. From now on, we fix a bidder α and an inefficient equilibrium $\mathbf{B}$, so that α is additive and $\alpha \in W(\mathbf{B})$.

Corollary 32 already gives us an insight about the competition in such an equilibrium $\mathbf{B}$. While bidder α will have to compete against the other bidders of $W(\mathbf{B})$ to win extra units, in addition to those that she is guaranteed to obtain, each bidder in $W(\mathbf{B}) \setminus \{\alpha\}$ only competes against α. Each of them is not guaranteed any units unless she outbids α (bidder α is the only cause of externality for bidders in $W(\mathbf{B}) \setminus \{\alpha\}$ and anyone bidding lower than α cannot get any units). If bidder α did not exist, the other winners could be automatically granted the demand they are requesting since, in total, it is smaller than k and hence, there is no competition among them.

Observation 34. $\hat{F}_i^{avg}(z) = F_\alpha(z)$, for every $i \in W(\mathbf{B}) \setminus \{\alpha\}$, where F_α is the CDF of bidder α.

To proceed, we identify some further properties on the support of the mixed strategies.

Lemma 35. For the equilibrium $\mathbf{B}$ under consideration, it is true that:

1. $Supp(B_\alpha) = [\ell(\mathbf{B}_W), h(\mathbf{B}_W)]$.

2. For any two bidders $i, j \in W(\mathbf{B}) \setminus \{\alpha\}$ such that $v_i \neq v_j$, the set $Supp(B_i) \cap Supp(B_j)$ is of measure 0 (intersection points can occur only at endpoints of intervals).

Lemma 35 suggests that we can group the bidders according to their values (since only bidders with the same value can overlap in their support). Let $r \leq |W(\mathbf{B}) \setminus \{\alpha\}|$ represent the number of distinct values $v_1, \ldots, v_r$ that bidders in $W(\mathbf{B}) \setminus \{\alpha\}$ have. We can partition the bidders of $W(\mathbf{B}) \setminus \{\alpha\}$ into r groups $W_1(\mathbf{B}), \ldots, W_r(\mathbf{B})$ such that, for $j = 1, \ldots, r$, the bidders in group $W_j(\mathbf{B})$ have value v_j. Similarly, we split the support of the winning bidders $[\ell(\mathbf{B}_W), h(\mathbf{B}_W)]$ into r intervals, i.e., $[\ell(\mathbf{B}_W), h(\mathbf{B}_W)] = \bigcup_{j=1}^{r} I_j(\mathbf{B})$, where each interval $j \in \{1, \ldots, r\}$ is formed as

$$I_j(\mathbf{B}) = \bigcup_{i \in W_j(\mathbf{B})} Supp(B_i)\,.$$

The following is a direct corollary of Lemma 35.

Corollary 36. For every $s, t \in \{1, \ldots, r\}$ with $s \neq t$, the set $I_s(\mathbf{B}) \cap I_t(\mathbf{B})$ is of measure 0.

When all bidders in $W(\mathbf{B}) \setminus \{\alpha\}$ have distinct values there are precisely $|W(\mathbf{B}) \setminus \{\alpha\}|$ intervals whereas when they all have a common value, they must be bidding on the entire interval $[\ell(W(\mathbf{B})), h(W(\mathbf{B}))]$ (the equilibrium in the 2-bidder case when $d_1 = k$, in Section 4.1, is one such example). We sometimes denote as $I_0(\mathbf{B})$ the interval of losing bidders $[0, \ell(\mathbf{B}_W)]$, i.e., for the bidders in $\mathcal{N} \setminus W(\mathbf{B})$. Note that given $\mathbf{B}$, the only criterion for the membership of the support of a bidder i in an interval $I_s(\mathbf{B})$ is their value.

The next step is quite crucial in simplifying the extraction of our upper bound. We show that the worst case demand structure for the bidders in $W(\mathbf{B}) \setminus \{\alpha\}$ is when they all have unit demand.

Theorem 37. For the value profile $(\mathbf{v}, \mathbf{d})$ and the equilibrium $\mathbf{B}$ under consideration, there exists another value profile $(\mathbf{v}', \mathbf{d}')$ and a product distribution $\mathbf{B}'$ such that

1. $\alpha \in W(\mathbf{B}')$ is an additive bidder and for every bidder $i \in W(\mathbf{B}') \setminus \{\alpha\}$, it holds that $d'_i = 1$.
2. $\mathbf{B}'$ is a mixed Nash equilibrium for $(\mathbf{v}', \mathbf{d}')$.
3. $\frac{OPT(\mathbf{v},\mathbf{d})}{SW(\mathbf{B})} = \frac{OPT(\mathbf{v}',\mathbf{d}')}{SW(\mathbf{B}')}$.

For the remainder of the section, it suffices to analyze valuation profiles, that possess equilibria where the members of $W(\mathbf{B})$ are either additive or unit-demand. Recall, that due to Corollary 32, there must be a unique additive bidder. Hence, we fix an instance given by a valuation profile $(\mathbf{v}, \mathbf{d})$, so that at the equilibrium $\mathbf{B}$, the set $W(\mathbf{B})$ consists of n unit-demand bidders plus the additive bidder α, i.e. $n = |W(\mathbf{B}) \setminus \{\alpha\}|$. Moreover, due to the following observation we may assume, without loss of generality, that the support of each unit-demand bidder has no overlapping intervals with other bidders from $W(\mathbf{B}) \setminus \{\alpha\}$.

Lemma 38. Let $(\mathbf{v}, \mathbf{d})$ be a value profile and let $\mathbf{B}$ be any mixed Nash equilibrium such that the members of $W(\mathbf{B})$ are all unit-demand bidders aside from one additive. Then, there exists a mixed Nash equilibrium $\mathbf{B}'$ with disjoint support intervals such that $SW(\mathbf{B}) = SW(\mathbf{B}')$.

Therefore, by Corollary 36 and the discussion preceding it, the support of each bidder $i = 1, \ldots, n$ is $[\ell(B_i), h(B_i)]$. Note that due to Lemma 35, the unit-demand bidders must cover the entire interval $[\ell(\mathbf{B}_W), h(\mathbf{B}_W)]$. Therefore, for a unit-demand bidder $i = 1, \ldots, n$, it must be that $\ell(B_i) = h(B_{i-1})$, assuming for convenience that $h(B_0) = \ell(\mathbf{B}_W)$.

We continue, by understanding further the support intervals and the distributions of the equilibrium $\mathbf{B}$.

Theorem 39. For the value profile $(\mathbf{v}, \mathbf{d})$ under consideration, the following properties hold:

1. For bidder α, we have

$$h(B_\alpha) = h(B_n) = h(\mathbf{B}_W) = v_\alpha - (k-n)\frac{v_\alpha - \ell(B_\alpha)}{k}\,.$$

   Moreover, for every unit-demand bidder $i = 1, \ldots, n-1$ it holds that

$$\ell(B_{i+1}) = h(B_i) = v_\alpha - \frac{(k-n)(v_\alpha - \ell(B_\alpha))}{k-n+i}\,. \tag{7}$$

2. The CDF F_α of bidder α, is a branch function, so that for $i = 1, \ldots, n$, $F_\alpha(z) = F^i_\alpha(z)$ for every $z \in [h(B_{i-1}), h(B_i)]$ with

$$F^i_\alpha(z) = \prod_{j=i+1}^{n} \left(\frac{v_j - h(B_j)}{v_j - h(B_{j-1})}\right) \frac{v_i - h(B_i)}{v_i - z}\,. \tag{8}$$

Proof. For the first part of the theorem, we can easily obtain the expression for $h(B_\alpha)$, for the additive bidder α, since she is the sole non-empty-handed bidder, by applying Lemma 24. To obtain the expressions for the rightmost points of the unit-demand bidders, we study the utility function of bidder α focusing on the points $h(B_1), h(B_2), \ldots, h(B_n)$. In fact, by Corollary 20 it must be that the expected utility at all these points is equal. Since these are the rightmost endpoints of the support of the unit-demand bidders (and none of them is a mass point for any of them), bidder α is guaranteed $i + k - n$ units when she bids $h(B_i)$. This means that for $i = 1, \ldots, n-1$,

$$\mathop{\mathbb{E}}_{\mathbf{b}_{-\alpha}\sim\mathbf{B}_{-\alpha}}[u_\alpha(h(B_i), \mathbf{b}_{-\alpha})] = \mathop{\mathbb{E}}_{\mathbf{b}_{-\alpha}\sim\mathbf{B}_{-\alpha}}[u_\alpha(h(B_{i+1}), \mathbf{b}_{-\alpha})] \Leftrightarrow$$
$$(k-n+i)(v_a - h(B_i)) = (k-n+i+1)(v_a - h(B(i+1)))\,.$$

The above yields a recursive relation, where $h(B_i)$ can be obtained as a function of $h(B_{i+1})$. Since $h(B_n) = h(B_a)$ is known, we can use induction and establish (7).

For the second part of the theorem, we use that for $i = 1, \ldots, n$, and $z \in Supp(B_i) = [h(B_{i-1}), h(B_i)]$, we have

$$\mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(z, \mathbf{b}_{-i})] = \mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(h(B_i), \mathbf{b}_{-i})] = F_\alpha(h(B_i))(v_i - h(B_i))\,,$$

where the last equality is by Observation 34, that $\hat{F}^{avg}_i(z) = F_\alpha(z)$. By using Theorem 22 for $\hat{F}^{avg}_i(z)$, we have

$$F_\alpha(z) = \frac{F_\alpha(h(B_i))(v_i - h(B_i))}{v_i - z} \ \ \forall z \in Supp(B_i)\,. \tag{9}$$

To proceed, it is convenient to think of F_α as a branch function, with a different branch corresponding to each support interval of the unit-demand bidders. In particular, we let $F^i_\alpha(z) = F_\alpha(z)$ for $z \in Supp(B_i)$. Moreover, by Lemma 19 the distribution F_α must have no mass points in its support at any intermediate point $h(B_i)$ for $i = 1, \ldots, n-1$. Therefore, since $h(B_i)$ belongs both to $Supp(B_i)$ and to $Supp(B_{i+1})$, in order to have continuity, it must hold that

$$F^i_\alpha(h(B_i)) = F^{i+1}_\alpha(h(B_i)) \ \ \forall i = 1, \ldots, n-1\,.$$

By combining the last two equalities, Equation (9) can be rewritten as

$$F^i_\alpha(z) = \frac{F^{i+1}_\alpha(h(B_i))(v_i - h(B_i))}{v_i - z}\,.$$

Hence, we have expressed F_α^i as dependent on the term $F_\alpha^{i+1}(h(B_i))$. Finally, since we know that $F_\alpha^n(h(B_n)) = 1$, we can work inductively and obtain the closed form of each branch F_{α_i}, which completes the proof. $\square$

Having described the structure of the equilibrium $\mathbf{B}$, we can compute its expected social welfare, which can be verified that it is given by the following expression.

Before stating our upper bound, we present two straightforward inequalities that are a direct consequence of the definition of a mixed Nash equilibrium. These inequalities will be very useful in obtaining the upper bound on the Price of Anarchy.

Lemma 40. Consider a value profile $(\mathbf{v}, \mathbf{d})$ and any inefficient mixed Nash Equilibrium $\mathbf{B}$ with a set $W(\mathbf{B})$ that consists only of additive or unit-demand bidders. The following properties hold

1. For $i = 1, \ldots, n-1$, $m = i+1, \ldots, n$ and every $z \in [h(B_{m-1}), h(B_m)]$ it holds that

$$\prod_{j=i+1}^{m-1} \frac{v_j - h(B_j)}{v_i - h(B_{j-1})} \geq \frac{v_i - z}{v_i - h(B_i)} \frac{v_m - h(B_{m-1})}{v_m - z} \,. \tag{10}$$

2. For $i = 2, \ldots, n$, $m = 1, \ldots, i-1$ and every $z \in [h(B_{m-1}), h(B_m)]$

$$\prod_{j=m+1}^{i-1} \frac{v_j - h(B_j)}{v_i - h(B_{j-1})} \leq \frac{v_m - z}{v_m - h(B_m)} \frac{v_i - h(B_{i-1})}{v_i - z} \,. \tag{11}$$

Proof. For bidder $i = 1, \ldots, n-1$, and $m = i+1, \ldots, n$ consider a unilateral deviation $z \in [h(B_{m-1}), h(B_m)]$ of bidder i. Then by the definition of a mixed Nash Equilibrium, Definition 1, it holds that

$$\mathop{\mathbb{E}}_{\mathbf{b}_{-i} \sim \mathbf{B}_{-i}}[u_i(z, \mathbf{b}_{-i})] \leq \mathop{\mathbb{E}}_{\mathbf{b} \sim \mathbf{B}}[u_i(\mathbf{b})] \Leftrightarrow F_{\alpha^m}(z)(v_i - z) \,.$$

By substituting the appropriate branches of F_α of Equation (8) the first inequality follow. The reasoning is identical for the second inequality, although this time a bidder $i = 2, \ldots, n$ examines a deviation to a lower interval $m = 1, \ldots, i-1$. $\square$

Lemma 41. Consider a value profile $(\mathbf{v}, \mathbf{d})$ and any inefficient mixed Nash Equilibrium $\mathbf{B}$ with a set $W(\mathbf{B})$ that consists only of additive or unit-demand bidders. The expected social welfare is

$$SW(\mathbf{B}) = kv_\alpha - (k-n)(v_\alpha - \ell(B_\alpha)) \sum_{i=1}^{n} \prod_{j=i+1}^{n} \left(\frac{v_j - h(B_j)}{v_j - h(B_{j-1})} \right) \int_{h(B_{i-1})}^{h(B_i)} \frac{v_i - h(B_i)}{v_i - z} \frac{v_\alpha - v_i}{(v_a - z)^2} dz \,. \tag{12}$$

Proof of Theorem 31. For brevity we denote $\ell(B_a)$ as ℓ and, for $j = 1, \ldots, n$, we denote $h(B_j)$ as h_j. Moreover, by assumption $v_a \geq v_n$. To simplify calculations, we assume that $v_a = 1$ by rescaling all values in the instance.

Given a mixed Nash equilibrium $\mathbf{B}$ we lower bound the expected social welfare $SW(\mathbf{B})$ described in Equation (12) as

$$\begin{aligned} SW(\mathbf{B}) &= k - (k-n)(1-\ell) \sum_{i=1}^{n} \prod_{j=i+1}^{n} \left(\frac{v_j - h_j}{v_j - h_{j-1}} \right) \int_{h_{i-1}}^{h_i} \frac{v_i - h_i}{v_i - z} \frac{1 - v_i}{(1-z)^2} dz \\ &= k - (k-n)(1-\ell) \sum_{i=1}^{n} \prod_{j=i+1}^{n} \left(\frac{v_j - h_j}{v_j - h_{j-1}} \right) \left(\int_{h_{i-1}}^{h_i} \frac{v_i - h_i}{v_i - z} \frac{1}{(1-z)} dz - \int_{h_{i-1}}^{h_i} \frac{v_i - h_i}{(1-z)^2} dz \right) \end{aligned}$$

$$
\begin{aligned}
&> k-(k-n)(1-\ell)\sum_{i=1}^{n}\prod_{j=i+1}^{n}\left(\frac{v_j-h_j}{v_j-h_{j-1}}\right)\int_{h_{i-1}}^{h_i}\frac{v_i-h_i}{v_i-z}\frac{1}{(1-z)}dz\\
&\geq k-(k-n)(1-\ell)\int_{\ell}^{h_n}\frac{v_n-h_n}{(v_n-z)(1-z)}dz \geq k-(k-n)(1-\ell)\int_{\ell}^{h_n}\frac{1-h_n}{(1-z)^2}dz\\
&\geq k-(k-n)(1-\ell)=k-(k-n)(h_n-\ell)=k-(k-n)\left(\frac{n}{k}(1-\ell)\right)\geq k-(k-n)\left(\frac{n}{k}\right)\\
&\geq \frac{3}{4}k\,.
\end{aligned}
$$

The first inequality is due to the fact that for all bidders $i = 1, \dots, n$, it holds that $v_i > h_i$ by Observation 16. The second is an application of the mixed Nash equilibrium property encoded by Equation (11) of Lemma 40. The next two inequalities occur by observing that the respective functions are increasing in terms of v_n (which, by assumption, we upper bound with $v_n \leq 1$) and ℓ (which we lower bound with $\ell \geq 0$). The last inequality follows by setting $x = \frac{n}{k}$ and minimizing the function $s(x) = 1 - x + x^2$ for $x \in (0, 1)$. The theorem follows by observing that the optimal welfare is k, since the additive bidder has the highest value.

□

5 A Separation Result between Mixed and Bayesian Price of Anarchy

In this section we move away from mixed Nash equilibria and explore the more general solution concept of the Bayes Nash equilibrium. We consider the following incomplete information setting. Let (v_i, d_i) be the type of bidder $i \in \mathcal{N}$. We suppose that the private value v_i of a bidder $i \in \mathcal{N}$ is drawn independently from a distribution V_i. The second part of bidder i's type is his demand d_i; for the purposes of this section (we only construct a lower bound instance), we assume d_i to be deterministic private information.

Each bidder i is aware of her own value per unit v_i and the product distribution $\times_j V_j$ and decides a strategy $(b_i, q_i) \sim G_i(v_i)$ for each value $v_i \sim V_i$. The bidding strategy is in general a mixed strategy. In the special case that bidder i chooses a single bid $(b_i(v_i), q_i)$ for each drawn value v_i, he submits a pure strategy, where q_i is not necessarily d_i.

Definition 42. Given $\mathbf{V} = \times_{i=1}^{n} V_i$ and $\mathbf{d}$, a profile $\mathbf{G}(\mathbf{v})$ is a Bayes Nash Equilibrium if for all $i \in \mathcal{N}$, v_i in V_i's domain, $b'_i \geq 0$ and $q'_i \in \mathbb{Z}_+$ it holds that

$$
\mathop{\mathbb{E}}_{\mathbf{v}_{-i}\sim\mathbf{V}_{-i}}\left[\mathop{\mathbb{E}}_{(\mathbf{b},\mathbf{q})\sim\mathbf{G}(\mathbf{v})}\left[u_i^{v_i}(\mathbf{b},\mathbf{q})\right]\right] \geq \mathop{\mathbb{E}}_{\mathbf{v}_{-i}\sim\mathbf{V}_{-i}}\left[\mathop{\mathbb{E}}_{(\mathbf{b}_{-i},\mathbf{q}_{-i})\sim\mathbf{G}_{-i}(\mathbf{v}_{-i})}\left[u_i^{v_i}((b'_i,q'_i),(\mathbf{b}_{-i},\mathbf{q}_{-i}))\right]\right], \tag{13}
$$

where $u_i^{v_i}(\cdot)$ stands for bidder i's utility when his value is v_i.

We can define the Bayesian Price of Anarchy in the same way as before, where now we compare against the expected optimal welfare, over the value distributions.

Although in a few other auction formats, the inefficiency does not get worse when one moves to incomplete information games, we exhibit that this is not the case here. We present a lower bound on the Bayesian Price of Anarchy of 1.1204, with two bidders. For mixed equilibria and two bidders, Theorem 25 showed that the Price of Anarchy is at most 1.1095. Although this difference is small, it shows that the Bayesian model is more expressive and can thus create more inefficiency. In particular, we stress that the bound obtained here for two bidders is inspired by the same bound of 1.1204 for mixed equilibria in Theorem 30 where we had to use a large number of bidders.

Theorem 43. For $n = 2$, $k \geq 2$, and capped additive valuation profiles, the Price of Anarchy of Bayes Nash equilibria is at least 1.1204.

Remark 44. When $k = 1$, there is a lower bound of 1.15 in [13] for the first price auction. However this requires a very large number of bidders. With $k = 1$ and two bidders, there is a simpler construction in [21] but it only yields a lower bound of 1.06.

References

[1] L. Ausubel and P. Cramton. Demand Reduction and Inefficiency in Multi-Unit Auctions. Technical report, University of Maryland, 2002.

[2] L. Ausubel, P. Cramton, M. Pycia, M. Rostek, and M. Weretka. Demand reduction and inefficiency in multi-unit auctions. *The Review of Economic Studies*, 81:1366–1400, 2014.

[3] B. Baisa and J. Burkett. Large multi-unit auctions with a large bidder. *J. Econ. Theory*, 174:1–15, 2018.

[4] K. Bhawalkar and T. Roughgarden. Welfare Guarantees for Combinatorial Auctions with Item Bidding. In *SODA*, pages 700–709. SIAM, 2011.

[5] G. Birmpas, E. Markakis, O. Telelis, and A. Tsikiridis. Tight welfare guarantees for pure nash equilibria of the uniform price auction. *Theory Comput. Syst.*, 63(7):1451–1469, 2019.

[6] M. Brenner, D. Galai, and O. Sade. Sovereign debt auctions: Uniform or discriminatory? *Journal of Monetary Economics*, 56(2):267–274, 2009.

[7] G. Christodoulou, A. Kovács, and M. Schapira. Bayesian combinatorial auctions. *J. ACM*, 63(2):11:1–11:19, 2016.

[8] G. Christodoulou, A. Kovács, A. Sgouritsa, and B. Tang. Tight Bounds for the Price of Anarchy of Simultaneous First-Price Auctions. *ACM TEAC*, 4(2):9:1–9:33, 2016.

[9] B. de Keijzer, E. Markakis, G. Schäfer, and O. Telelis. Inefficiency of Standard Multi-unit Auctions. In *ESA*, pages 385–396. Springer, 2013.

[10] M. Feldman, H. Fu, N. Gravin, and B. Lucier. Simultaneous Auctions are (almost) Efficient. In *STOC*, pages 201–210, 2013.

[11] M. Friedman. *A Program for Monetary Stability*. Fordham University Press, New York, NY, 1960.

[12] K. Goldner, N. Immorlica, and B. Lucier. Reducing inefficiency in carbon auctions with imperfect competition. In *ITCS*, pages 15:1–15:21, 2020.

[13] J. Hartline, D. Hoy, and S. Taggart. Price of anarchy for auction revenue. In *ACM Conference on Economics and Computation, EC 2014*, pages 693–710, 2014.

[14] V. Krishna. *Auction Theory*. Academic Press, 2002.

[15] P. Milgrom. *Putting Auction Theory to Work*. Cambridge University Press, 2004.

[16] A. Ockenfels, D. H. Reiley, and A. Sadrieh. *Economics and Information Systems*, volume 1 of *Handbooks in Information Systems*, chapter 12. Online Auctions, pages 571–628. 2006.

[17] M. Pycia and K. Woodward. Auctions of homogeneous goods: A case for pay-as-bid. In *ACM EC*, 2021. to appear.

[18] P. Del Rio. Designing auctions for renewable electricity support. best practices from around the world. *Energy for Sustainable Development*, 41:1–13, 2017.

[19] T. Roughgarden. The price of anarchy in games of incomplete information. In *ACM EC*, pages 862–879, 2012.

[20] J. Swinkels. Efficiency of large private value auctions. *J. Econ. Theory*, 69(1):37–68, 2001.

[21] V. Syrgkanis. *Efficiency of Mechanisms in Complex Markets*. PhD thesis, Cornell University, 2014.

[22] V. Syrgkanis and É. Tardos. Composable and efficient mechanisms. In *STOC*, pages 211–220, 2013.

[23] W. Vickrey. Counterspeculation, Auctions, and Competitive Sealed Tenders. *Journal of Finance*, 16(1):8–37, 1961.

A Missing proofs from Section 3.1

Proof of Lemma 4. We use first the following auxiliary lemma.

Lemma 45. Let $\mathbf{G}$ be any mixed Nash equilibrium where there exists a bidder i such that $Pr[b_i = v_i, q_i < d_i] > 0$ for $(\mathbf{b}, \mathbf{q}) \sim \mathbf{G}$. Let G'_i be the same as G_i after replacing any $q_i < d_i$ with d_i. Then $\mathbf{G}'$ is also a mixed Nash equilibrium with the same social welfare.

Proof. First note that $\mathbb{E}_{(\mathbf{b},\mathbf{q})\sim\mathbf{G}}[u_i(\mathbf{b}, \mathbf{q})] = 0$, since bidder i bids v_i with positive probability that results in zero utility (see also Lemma 11). Let β^k_{-i} be the random variable expressing the k^{th} maximum payment under $\mathbf{G}_{-i}$. Then $Pr[\beta^k_{-i} < v_i] = 0$, because if β^k_{-i} takes a value less than v_i with positive probability, bidder i has an incentive to deviate to a bid less than v_i and receive positive utility.

For any bidder j, with $v_j > v_i$, and any bidding profile $(\mathbf{b}, \mathbf{q}) \sim \mathbf{G}$, such that $x_i(\mathbf{b}, \mathbf{q}) > 0$ and $x_j(\mathbf{b}, \mathbf{q}) < q_j$, it holds that $b_j \leq b_i = v_i$ (apart maybe from cases that appear with zero probability). Then, $Pr[x_i(\mathbf{b}, \mathbf{q}] > 0, x_j(\mathbf{b}, \mathbf{q}) < q_j) = 0$, otherwise there exists a sufficiently small $\varepsilon > 0$, such that bidder j has an incentive to deviate to $v_i + \varepsilon$ and receive more units. Therefore, if bidders i and j bid both v_i with positive probability the tie-breaking rule is in favour of player j. The same tie-breaking rule should be applied when bidder i increases his quantity bid and so, for any bidding profile $(\mathbf{b}, \mathbf{q}) \sim \mathbf{G}$, $x_j(\mathbf{b}, \mathbf{q}) = x_j(\mathbf{b}, (d_i, \mathbf{q}_{-i}))$.

For any bidder j, with $v_j \leq v_i$, bidder j cannot get any unit by paying less that v_i since $Pr[\beta^k_{-i} < v_i] = 0$. Therefore bidder j may receive units with positive probability only if $v_j = v_i$ and his expected utility is zero.

Overall, if bidder i deviates from G_i to G'_i, either the allocation of the players remains the same (so they still have no incentive to deviate) or they have zero utility (and still no incentive to deviate) and the allocation may change between bidders of the same valuation; so the expected social welfare remains the same and the new strategy profile is a mixed Nash equilibrium.

□

We continue now with the proof of Lemma 4. Starting by $\mathbf{G}$, we recursively show that if one by one the bidders deviate to G'_i, the bidding profile remains an equilibrium with the same social welfare. It is sufficient to show this for $(G'_i, \mathbf{G}_{-i})$.

First note that, according to the tie-breaking rule, if i deviates from G_i to G'_i, he can only get more units as he only declares the same or more demand. Let S_i be the set of bids (b_i, q_i) such that $q_i < d_i$ and

$$\mathop{\mathbb{E}}_{(\mathbf{b}_{-i},\mathbf{q}_{-i})\sim\mathbf{G}_{-i}}[x_i((b_i, \mathbf{b}_{-i}), (q_i, \mathbf{q}_{-i}))] < \mathop{\mathbb{E}}_{(\mathbf{b}_{-i},\mathbf{q}_{-i})\sim\mathbf{G}_{-i}}[x_i((b_i, \mathbf{b}_{-i}), (d_i, \mathbf{q}_{-i}))]$$

It should be that for $(\mathbf{b}, \mathbf{q}) \sim \mathbf{G}$, $Pr[(b_i, q_i) \in S_i, b_i < v_i] = 0$, otherwise bidder i would increase her utility by deviating from $(b_i, q_i) \in S_i$ to (b_i, d_i). So, the only case that bidder i may increase his allocation by increasing his demand to d_i is when he bids his value, in which case the lemma follows by Lemma 45.

So far we have shown that for any bid (b_i, q_i) such that $q_i < d_i$ and $b_i < v_i$ the expected allocation to bidder i remains the same if he deviates to (b_i, d_i), apart maybe from cases that appear with zero probability. It remains to show that under $(G'_i, \mathbf{G}_{-i})$ the allocation to all bidders remains the same. Note that i deviating from G_i to G'_i can only cause other bidders to be allocated less or the same number of units due to the tie breaking rule. Given any $(b_i, q_i) \sim G_i$ with $q_i < d_i$, let $S_{-i}(b_i, q_i) = S_{-i}$ be the set of bidding profiles $(\mathbf{b}_{-i}, \mathbf{q}_{-i}) \sim \mathbf{G}_{-i}$ such that there exists a bidder $j \neq i$, receiving less units by the deviation of i, i.e., $x_j(\mathbf{b}, \mathbf{q}) > x_j(\mathbf{b}, (d_i, \mathbf{q}_{-i}))$, where $\mathbf{b} = (b_i, \mathbf{b}_{-i})$, and $\mathbf{q} = (q_i, \mathbf{q}_{-i})$.

For the sake of contradiction suppose that, under $\mathbf{G}_{-i}$, $Pr[(\mathbf{b}_{-i}, \mathbf{q}_{-i}) \in S_{-i}] > 0$. By summing over all bidders but i and taking the expectation over $\mathbf{G}_{-i}$, we have that

$$\mathop{\mathbb{E}}_{(\mathbf{b}_{-i},\mathbf{q}_{-i})\sim\mathbf{G}_{-i}}[\sum_{j\neq i} x_j(\mathbf{b}, \mathbf{q})] > \mathop{\mathbb{E}}_{(\mathbf{b}_{-i},\mathbf{q}_{-i})\sim\mathbf{G}_{-i}}[\sum_{j\neq i} x_j(\mathbf{b}, (d_i, \mathbf{q}_{-i}))] \,.$$

This means that $\mathbb{E}_{(\mathbf{b}_{-i},\mathbf{q}_{-i})\sim\mathbf{G}_{-i}}[x_i(\mathbf{b},\mathbf{q})] < \mathbb{E}_{(\mathbf{b}_{-i},\mathbf{q}_{-i})\sim\mathbf{G}_{-i}}[x_i(\mathbf{b},(d_i,\mathbf{q}_{-i}))]$ which leads to a contradiction.

□

Proof of Lemma 5. The proof is established by the following two lemmas.

Lemma 46. For any Nash equilibrium $\mathbf{G}$ where nobody declares less demand, if $Pr[x_i(\mathbf{b},\mathbf{q}) > d_i] > 0$ for $(\mathbf{b},\mathbf{q}) \sim \mathbf{G}$, then $\mathbb{E}[b_i \mid x_i(\mathbf{b},\mathbf{q}) > d_i] = 0$.

Proof. Suppose on the contrary that $\mathbb{E}[b_i \mid x_i(\mathbf{b},\mathbf{q}) > d_i] > 0$. We will show that bidder i has an incentive to declare her true demand instead of a higher demand.

$$\begin{aligned}
\mathbb{E}[u_i(\mathbf{b},\mathbf{q})] &= Pr[x_i(\mathbf{b},\mathbf{q}) > d_i]\,\mathbb{E}[u_i(\mathbf{b},\mathbf{q}) \mid x_i(\mathbf{b},\mathbf{q}) > d_i] + Pr[x_i(\mathbf{b},\mathbf{q}) \le d_i]\,\mathbb{E}[u_i(\mathbf{b},\mathbf{q}) \mid x_i(\mathbf{b},\mathbf{q}) \le d_i] \\
&= Pr[x_i(\mathbf{b},\mathbf{q}) > d_i]\,\mathbb{E}[d_i(v_i-b_i) - (x_i(\mathbf{b},\mathbf{q})-d_i)b_i \mid x_i(\mathbf{b},\mathbf{q}) > d_i] \\
&\quad + Pr[x_i(\mathbf{b},\mathbf{q}) \le d_i]\,\mathbb{E}[x_i(\mathbf{b},\mathbf{q})(v_i-b_i) \mid x_i(\mathbf{b},\mathbf{q}) \le d_i] \\
&= Pr[x_i(\mathbf{b},\mathbf{q}) > d_i]\,\mathbb{E}[d_i(v_i-b_i) \mid x_i(\mathbf{b},\mathbf{q}) > d_i] \\
&\quad + Pr[x_i(\mathbf{b},\mathbf{q}) \le d_i]\,\mathbb{E}[x_i(\mathbf{b},\mathbf{q})(v_i-b_i) \mid x_i(\mathbf{b},\mathbf{q}) \le d_i] \\
&\quad - Pr[x_i(\mathbf{b},\mathbf{q}) > d_i]\,\mathbb{E}[(x_i(\mathbf{b},\mathbf{q})-d_i)b_i \mid x_i(\mathbf{b},\mathbf{q}) > d_i] \\
&= \mathbb{E}[u_i(\mathbf{b},d_i,\mathbf{q}_{-i})] - Pr[x_i(\mathbf{b},\mathbf{q}) > d_i]\,\mathbb{E}[(x_i(\mathbf{b},\mathbf{q})-d_i)b_i \mid x_i(\mathbf{b},\mathbf{q}) > d_i] \\
&\le \mathbb{E}[u_i(\mathbf{b},d_i,\mathbf{q}_{-i})] - Pr[x_i(\mathbf{b},\mathbf{q}) > d_i]\,\mathbb{E}[b_i \mid x_i(\mathbf{b},\mathbf{q}) > d_i] \\
&< \mathbb{E}[u_i(\mathbf{b},d_i,\mathbf{q}_{-i})]\,,
\end{aligned}$$

where the last strict inequality is due to our assumption that $\mathbb{E}[b_i \mid x_i(\mathbf{b},\mathbf{q}) > d_i] > 0$. The lemma follows by contradiction. □

Lemma 47. If $\sum_i d_i > k$ then in any Nash equilibrium $\mathbf{G}$ where nobody declares less demand, $Pr[x_i(\mathbf{b},\mathbf{q}) > d_i] = 0$ for all i where $(\mathbf{b},\mathbf{q}) \sim \mathbf{G}$.

Proof. Suppose on the contrary that there exists a bidder i such that $Pr[x_i(\mathbf{b},\mathbf{q}) > d_i] > 0$. Then there is also a bidder j such that $Pr[x_i(\mathbf{b},\mathbf{q}) > d_i, x_j(\mathbf{b},\mathbf{q}) < d_j] > 0$, otherwise $Pr[x_j(\mathbf{b},\mathbf{q}) \ge d_j, \forall j] = 1$ which contradicts the fact that $\sum_i d_i > k$.

Given that $x_i(\mathbf{b},\mathbf{q}) > d_i$ and $x_j(\mathbf{b},\mathbf{q}) < d_j$, bidder j bids 0 (apart maybe for cases that appear with zero probability), otherwise he should have received more units and bidder i less units since by Lemma 46, bidder i bids 0 and receives at least one unit. Then the expected utility of bidder j can be expressed as:

$$\begin{aligned}
\mathbb{E}[u_j(\mathbf{b},\mathbf{q})] &= Pr[x_i(\mathbf{b},\mathbf{q}) > d_i, x_j(\mathbf{b},\mathbf{q}) < d_j]\,\mathbb{E}[x_j(\mathbf{b},\mathbf{q})(v_j-b_j) \mid x_i(\mathbf{b},\mathbf{q}) > d_i, x_j(\mathbf{b},\mathbf{q}) < d_j] \\
&\quad + (1 - Pr[x_i(\mathbf{b},\mathbf{q}) > d_i, x_j(\mathbf{b},\mathbf{q}) < d_j])\,\mathbb{E}[u_j(\mathbf{b},\mathbf{q}) \mid x_i(\mathbf{b},\mathbf{q}) \le d_i \text{ or } x_j(\mathbf{b},\mathbf{q}) \ge d_j] \\
&\le Pr[x_i(\mathbf{b},\mathbf{q}) > d_i, x_j(\mathbf{b},\mathbf{q}) < d_j]\,\mathbb{E}[x_j(\mathbf{b},\mathbf{q})v_j \mid x_i(\mathbf{b},\mathbf{q}) > d_i, x_j(\mathbf{b},\mathbf{q}) < d_j] \\
&\quad + (1 - Pr[x_i(\mathbf{b},\mathbf{q}) > d_i, x_j(\mathbf{b},\mathbf{q}) < d_j])\,\mathbb{E}[u_j(\mathbf{b},\mathbf{q}) \mid x_i(\mathbf{b},\mathbf{q}) \le d_i \text{ or } x_j(\mathbf{b},\mathbf{q}) \ge d_j]\,.
\end{aligned}$$

Consider now the bidding strategy (b'_j, q_j) where $b'_j = \varepsilon > 0$ when $b_j = 0$ and $b'_j = b_j$ otherwise, for some $\varepsilon < Pr[x_i(\mathbf{b},\mathbf{q}) > d_i, x_j(\mathbf{b},\mathbf{q}) < d_j]v_j/k$. If bidder j deviates to this strategy he should receive at least one more unit since he would bid more than bidder i and his expected utility would be:

$$
\begin{aligned}
\mathbb{E}[u_j(b'_j, \mathbf{b}_{-j}, \mathbf{q})] &= Pr[x_i(\mathbf{b},\mathbf{q}) > d_i, x_j(\mathbf{b},\mathbf{q}) < d_j]\, \mathbb{E}[x_j(b'_j, \mathbf{b}_{-j}, \mathbf{q})(v_j - b'_j) \mid x_i(\mathbf{b},\mathbf{q}) > d_i, x_j(\mathbf{b},\mathbf{q}) < d_j] \\
&\quad + (1 - Pr[x_i(\mathbf{b},\mathbf{q}) > d_i, x_j(\mathbf{b},\mathbf{q}) < d_j])\, \mathbb{E}[u_j(b'_j, \mathbf{b}_{-j}, \mathbf{q}) \mid x_i(\mathbf{b},\mathbf{q}) \le d_i \text{ or } x_j(\mathbf{b},\mathbf{q}) \ge d_j] \\
&\ge Pr[x_i(\mathbf{b},\mathbf{q}) > d_i, x_j(\mathbf{b},\mathbf{q}) < d_j]\, \mathbb{E}[(x_j(\mathbf{b},\mathbf{q}) + 1)(v_j - \varepsilon) \mid x_i(\mathbf{b},\mathbf{q}) > d_i, x_j(\mathbf{b},\mathbf{q}) < d_j] \\
&\quad + (1 - Pr[x_i(\mathbf{b},\mathbf{q}) > d_i, x_j(\mathbf{b},\mathbf{q}) < d_j])\, \mathbb{E}[u_j(\mathbf{b},\mathbf{q}) - k\varepsilon \mid x_i(\mathbf{b},\mathbf{q}) \le d_i \text{ or } x_j(\mathbf{b},\mathbf{q}) \ge d_j] \\
&\ge \mathbb{E}[u_j(\mathbf{b},\mathbf{q})] + Pr[x_i(\mathbf{b},\mathbf{q}) > d_i, x_j(\mathbf{b},\mathbf{q}) < d_j](v_j - d_j\varepsilon) \\
&\quad - (1 - Pr[x_i(\mathbf{b},\mathbf{q}) > d_i, x_j(\mathbf{b},\mathbf{q}) < d_j])k\varepsilon \\
&\ge \mathbb{E}[u_i(\mathbf{b},\mathbf{q})] + Pr[x_i(\mathbf{b},\mathbf{q}) > d_i, x_j(\mathbf{b},\mathbf{q}) < d_j]v_j - k\varepsilon \\
&> \mathbb{E}[u_i(\mathbf{b},\mathbf{q})]\,,
\end{aligned}
$$

where the strict inequality comes from the definition of ε. This leads to a contradiction that concludes the proof. □

□

Proof of Lemma 6. If $\sum_i d_i \le k$, the optimum allocation appears when every bidder with positive valuation receives a number of units more or equal to their true demand.

For the sake of contradiction suppose that there exists a Nash equilibrium $\mathbf{G}$ and a bidder i such that $Pr[x_i(\mathbf{b},\mathbf{q}) < d_i] > 0$ for $(\mathbf{b},\mathbf{q}) \sim \mathbf{G}$. Since nobody bids less than their true demand, bidder i receives less units than d_i only because there are either bidders bidding more than their true demand or bidders with zero valuation receive units (or both). By Lemma 46, we have that $Pr[\max_{j \ne i : x_j(\mathbf{b},\mathbf{q}) > d_j} b_i = 0] = 0$ and the expected utility of bidder i can be expressed as follows:

$$
\begin{aligned}
\mathbb{E}[u_i(\mathbf{b},\mathbf{q})] &= Pr[x_i(\mathbf{b},\mathbf{q}) < d_i]\, \mathbb{E}[x_i(\mathbf{b},\mathbf{q})(v_i - b_i) \mid x_i(\mathbf{b},\mathbf{q}) < d_i, \max_{j \ne i, x_j(\mathbf{b},\mathbf{q}) > d_j} b_j = 0] \\
&\quad + Pr[x_i(\mathbf{b},\mathbf{q}) \ge d_i]\, \mathbb{E}[d_i v_i - x_i(\mathbf{b},\mathbf{q}) b_i \mid x_i(\mathbf{b},\mathbf{q}) \ge d_i] \\
&\le Pr[x_i(\mathbf{b},\mathbf{q}) < d_i]\, \mathbb{E}[(d_i - 1)v_i \mid x_i(\mathbf{b},\mathbf{q}) < d_i, \max_{j \ne i, x_j(\mathbf{b},\mathbf{q}) > d_j} b_j = 0] \\
&\quad + Pr[x_i(\mathbf{b},\mathbf{q}) \ge d_i]\, \mathbb{E}[d_i v_i - x_i(\mathbf{b},\mathbf{q}) b_i \mid x_i(\mathbf{b},\mathbf{q}) \ge d_i]\,,
\end{aligned}
$$

where the inequality is due to the fact that in the first term $x_i(\mathbf{b},\mathbf{q}) \le d_i - 1$; further note that in the first term bidder i loses units where the maximum of the other bids is 0, so he should have bid 0.

Consider now the bidding strategy (b'_i, q_i) where $b'_i = \varepsilon > 0$ when $b_i = 0$ and $b'_i = b_i$ otherwise, for some $\varepsilon < Pr[x_i(\mathbf{b},\mathbf{q}) < d_i]v_i/k$. Then the expected utility of bidder i after deviating to this strategy is:

$$
\begin{aligned}
\mathbb{E}[u_i(b'_i, \mathbf{b}_{-i}, \mathbf{q})] &= Pr[x_i(\mathbf{b},\mathbf{q}) < d_i]\, \mathbb{E}[x_i(b'_i, \mathbf{b}_{-i}, \mathbf{q})(v_i - b'_i) \mid x_i(\mathbf{b},\mathbf{q}) < d_i, \max_{j \ne i, x_j(\mathbf{b},\mathbf{q}) > d_j} b_j = 0] \\
&\quad + Pr[x_i(\mathbf{b},\mathbf{q}) \ge d_i]\, \mathbb{E}[d_i v_i - x_i(b'_i, \mathbf{b}_{-i}, \mathbf{q}) b'_i \mid x_i(\mathbf{b},\mathbf{q}) \ge d_i] \\
&\ge Pr[x_i(\mathbf{b},\mathbf{q}) < d_i]\, \mathbb{E}[d_i(v_i - \varepsilon) \mid x_i(\mathbf{b},\mathbf{q}) < d_i, \max_{j \ne i, x_j(\mathbf{b},\mathbf{q}) > d_j} b_j = 0] \\
&\quad + Pr[x_i(\mathbf{b},\mathbf{q}) \ge d_i]\, \mathbb{E}[d_i v_i - x_i(\mathbf{b},\mathbf{q})(b_i + \varepsilon) \mid x_i(\mathbf{b},\mathbf{q}) \ge d_i] \\
&\ge \mathbb{E}[u_i(\mathbf{b},\mathbf{q})] + Pr[x_i(\mathbf{b},\mathbf{q}) < d_i](v_i - d_i\varepsilon) - Pr[x_i(\mathbf{b},\mathbf{q}) \ge d_i]k\varepsilon \\
&\ge \mathbb{E}[u_i(\mathbf{b},\mathbf{q})] + Pr[x_i(\mathbf{b},\mathbf{q}) < d_i]v_i - k\varepsilon \\
&> \mathbb{E}[u_i(\mathbf{b},\mathbf{q})]\,,
\end{aligned}
$$

where the strict inequality comes from the definition of ε. This leads to a contradiction that concludes the proof. □

B Missing proofs from Section 3.2

Proof of Lemma 9. Let i be such a bidder. Since i receives at least one unit with positive probability, it holds that $Pr[x_i(\mathbf{b}) > 0] > 0$ for $\mathbf{b} \sim \mathbf{B}$. There is only one possible case so that bidder i has zero expected utility and this is that he bids his valuation when he receives at least one unit (or more accurately, the probability that he bids less than his value and receives at least one unit is zero).

First note that the payment for any unit is at least v_i (apart maybe from cases that appear with zero probability), otherwise bidder i has an incentive to bid less than v_i and get a positive utility. For any bidder j with $v_j > v_i$, $Pr[x_i(\mathbf{b}) > 0, x_j(\mathbf{b}) < d_j] = 0$, otherwise there exists a sufficiently small $\varepsilon > 0$, such that bidder j has an incentive to deviate from v_i to $v_i + \varepsilon$ and receive more units. Therefore, it holds that

$$Pr[x_j(\mathbf{b}) = d_j, \ \ \forall j \text{ with } v_j > v_i \mid x_i(\mathbf{b}) > 0] = 1,$$

and since $Pr[x_i(\mathbf{b}) > 0] > 0$ it holds that

$$Pr[x_j(\mathbf{b}) = d_j, \ \ \forall j \text{ with } v_j > v_i] > 0.$$

Since there are allocations where all bidders with higher valuation that v_i receive their demand, it is $\sum_{j:v_j>v_i} d_j < k$. Moreover, whenever bidders with valuation at most v_i receive units (these bidders must have zero expected utility since the lower payment is v_i), those bidders with valuation higher than v_i receive their demand. Overall, bidders with valuation higher than v_i receive their demand with probability 1. The rest of the units are given to bidders with valuation v_i (because the payment is at least v_i) which leads to optimal social welfare. □

Proof of Lemma 10. Suppose on the contrary that there exists only a single bidder i with $\mathbb{E}_{\mathbf{b}\sim B}[u_i(\mathbf{b})] > 0$ and for any other bidder $j \neq i$, $\mathbb{E}_{\mathbf{b}\sim B}[u_j(\mathbf{b})] = 0$. By Lemma 9, $\mathbb{E}_{\mathbf{b}\sim B}[x_j(\mathbf{b})] = 0$, for all $j \neq i$ and therefore, $\mathbb{E}_{\mathbf{b}\sim B}[x_i(\mathbf{b})] = k$. Moreover, since $\mathbf{B}$ is inefficient there exists a bidder $i' \neq i$ with $v_{i'} > v_i$. Since i has a positive expected utility, he receives the units in a price less than v_i. If bidder i' bids v_i, he can satisfy his demand which results in a positive expected utility leading to a contradiction. □

Proof of Observation 12. First note that $z < v_i$, otherwise, by Fact 11, $\mathbb{E}_{\mathbf{b}\sim \mathbf{B}}[u_i(\mathbf{b})] = 0$ and $i \notin W(\mathbf{B})$ which is a contradiction. The same holds for bidder j.

Since it cannot be the case that the tie-breaking rule always favors both bidders, either bidder i or bidder j (or both) can find a small enough δ so that transferring all the mass from z to $z + \delta$ will yield higher utility. □

Proof of Theorem 13. If z is a mass point for bidder i, then we are done by Fact 11. If not, then consider an interval $I \subseteq Supp(B_i)$ with $z \in I$ where nobody has a mass point in it (recall that the other bidders have no mass point on z, so such I exists). We analyze first the expected utility of a bidder i, given that she bids in I:

$$\begin{aligned}
\mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b}) \mid b_i \in I] &= \mathop{\mathbb{E}}_{b_i\sim B_i}[\mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(b_i, \mathbf{b}_{-i})] \mid b_i \in I] \\
&= \int_{z\in I} f_{b_i|b_i\in I}(z) \mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(z, \mathbf{b}_{-i})]dz \\
&= \frac{1}{Pr[b_i \in I]} \int_{z\in I} f_i(z) \mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(z, \mathbf{b}_{-i})]dz \qquad (14)
\end{aligned}$$

where f_i is the pdf of B_i, and $f_{b_i|b_i\in I}$ is the conditional pdf when $b_i \in I$. Note that for all $z \in I$ it holds that $f_i(z) \geq 0$ and f_i is continuous. Since no bidder has a mass point in I, by Fact 7 and Remark 8 it holds that $\mathbb{E}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(z, \mathbf{b}_{-i})]$ is also continuous in I as a function of z.

We now use a standard fact from calculus, commonly referred to as the integral version of the mean value theorem.

Fact 48. Let f, g be continuous functions on $[a, b]$ such that f is non-negative. Then there exists a $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)dx = g(c)\int_a^b f(x)dx.$$

Using this fact, we get that there exists $\xi \in I$, so that we can write Equation (14) as

$$\mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b}) \mid b_i \in I] = \mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(\xi, \mathbf{b}_{-i})]\frac{\int_{z\in I} f_i(z)dz}{Pr[b_i \in I]} = \mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(\xi, \mathbf{b}_{-i})]$$

Let $u = \mathbb{E}_{\mathbf{b}\sim B}[u_i(\mathbf{b})]$. By Fact 11, what we have established so far is that there exists a $\xi \in I$ for which

$$\mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(\xi, \mathbf{b}_{-i})] = u.$$

Consider now making the interval I smaller and smaller, by taking a sequence $I_1, I_2, \ldots$ such that in the limit, I_k collapses to z as $k \to \infty$. By the previous arguments, for every I_k, there exists a ξ_k such that $\mathbb{E}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(\xi_k, \mathbf{b}_{-i})] = u$. In the limit, $\xi_k \to z$ and we obtain that $\mathbb{E}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(z, \mathbf{b}_{-i})] = u$. □

C Missing proofs from Section 3.3

Proof of Lemma 17. Suppose for contradiction that i is a non-empty-handed bidder, and there exists a bidder $j \in W(\mathbf{B}) \setminus \{i\}$ (non-empty-handed or not), such that $h(B_j) > h(B_i)$. Since $j \in W(\mathbf{B})$, it must be that $v_j \geq h(B_j)$ and bidder j obtains positive utility when she bids in $Supp(B_j) \cap (h(B_i), h(B_j)]$ (otherwise j would have an incentive not to bid above $h(B_i)$). Moreover, $\mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[x_j(\mathbf{b}) \mid b_i \in (h(B_i), h(B_j)]] = d_j$, since outbidding a non-empty-handed bidder guarantees the allocation of a bidder's entire demand by the auction. However, bidder j can then benefit from transferring probability mass from $(h(B_i), h(B_j)]$ to a point $h(B_i) + \delta$, for some small enough $\delta > 0$, since it still guarantees the allocation of her entire demand but for a strictly better price and thus strictly better expected utility. Hence, we have proved that $h(B_j) \leq h(B_i)$.

We will now also prove that $h(B_i) \leq \max_{j\in W(\mathbf{B})\setminus\{i\}} h(B_j)$. Consider a bidder $j \in W(\mathbf{B}) \setminus \{i\}$. If j is also non-empty-handed, then we can just repeat the argument above by switching the places of i and j, and we are done. Otherwise, i is the only non-empty-handed bidder and suppose $h(B_i) > h(B_j)$ for every $j \in W(\mathbf{B}) \setminus \{i\}$. Then, whenever bidder i bids above every $h(B_j)$, she ranks first, and hence she is granted all her demand. But then, she has incentives to reduce her bid so that she is still above every $h(B_j)$ and win the same units at a lower price, which is a contradiction. So, $h(B_i) = h(\mathbf{B}_{W\setminus\{i\}})$. Then it is straightforward to see that $h(\mathbf{B}_{W\setminus\{i\}}) = h(\mathbf{B}_W)$. □

Proof of Lemma 18. Fix a bidder i and let $I \subseteq Supp(B_i)$ be any interval such that $I \nsubseteq \bigcup_{j\in W(\mathbf{B})\setminus\{i\}} Supp(B_j)$. We distinguish two cases: either $I = [\ell, h]$ for $\ell < h$, or I is an isolated point.

For the first case we can establish that i would have an incentive to bid only on ℓ and still win the same units at a lower price. Indeed,

$$\begin{aligned}
\mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b}) \mid b_i \in I] &= \mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}}[x_i(b_i, \mathbf{b}_{-i})(v_i - b_i) \mid b_i \in I] \\
&= \mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}}[x_i(\ell, \mathbf{b}_{-i})(v_i - b_i) \mid b_i \in I] \\
&< \mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[x_i(\ell, \mathbf{b}_{-i})(v_i - \ell)]
\end{aligned}$$

The second equality is due to the fact that no other bidder in $W(\mathbf{B})$ bids in I with positive probability, whereas the strict inequality follows by the construction of I. This inequality yields a contradiction, by Fact 11, since there exists a profitable transfer of the probability mass of I to the point ℓ.

For the second case suppose that I is some isolated point z that i bids with positive probability; z is isolated because we have assumed WLOG that all intervals defining the domain of a distribution are closed. Let $h = \max_{j\in W(\mathbf{B})\setminus\{i\}} h(B_j)$. If $z > h$, then i would be benefited by transferring the probability of bidding z to any point between h and z. If $z < h$, let z' be the maximum point such that $[z, z') \nsubseteq \bigcup_{j\in W(\mathbf{B})\setminus\{i\}} Supp(B_j)$ (note that $z \neq z'$). Then, there exists a bidder $i' \neq i$ such that, by Theorem 13, $\mathbb{E}_{\mathbf{b}_{-i'}\sim B_{-i'}}[u_{i'}(z', \mathbf{b}_{-i'})] = \mathbb{E}_{\mathbf{b}\sim B}[u_{i'}(\mathbf{b})]$. Bidding any bid between z and z' would result to a higher expected utility for bidder i' than $\mathbb{E}_{\mathbf{b}_{-i'}\sim B_{-i'}}[u_{i'}(z', \mathbf{b}_{-i'})]$, which is a contradiction to the fact that $\mathbf{B}$ is a Nash equilibrium. $\square$

Proof of Lemma 19. Regarding the first part of the statement, suppose, for contradiction, that there exists a bidder $i \in W(\mathbf{b})$, and a point $z \in Supp(B_i) \setminus \{\ell(\mathbf{B}_W)\}$ with $F_i(z) > \lim_{z\to z^-} F_i(z)$. By Observation 12 there exists no other bidder $j \in W(\mathbf{B}) \setminus \{i\}$ who also bids z with positive probability. We next distinguish the following cases:

Case 1: There does not exist any bidder $j \in W(\mathbf{B} \setminus \{i\})$ with an interval $I = [z-\delta, z] \subset Supp(B_j)$ for some small enough $\delta > 0$. Then by Lemma 18 it must also be that the interval $[z-\delta, z)$ is not in the support of bidder i. Furthermore, I is not in the support of any other bidder, who do not belong to $W(\mathbf{B})$ (since $z \neq \ell(\mathbf{B}_W)$, bidders not in $W(\mathbf{B})$ cannot use an interval of this form, because then they would have a positive probability of winning). Thus, bidder i could just choose a bid $\xi \in I$ with $\xi < z$ and win the same units as when bidding z at a lower price, which is a contradiction of $\mathbf{B}$ being an equilibrium.

Case 2: There exists a bidder $j \in W(\mathbf{B} \setminus \{i\})$ with an interval $I = [z-\delta, z] \subset Supp(B_j)$ for some small enough $\delta > 0$. But then

$$
\begin{aligned}
\mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}}[u_j(b_j, \mathbf{b}_{-j}) \mid b_j \in [z-\delta, z]] &= \mathop{\mathbb{E}}_{b_j\sim B_j}\big[\mathop{\mathbb{E}}_{\mathbf{b}_{-j}\sim\mathbf{B}_{-j}}[u_j(b_j, \mathbf{b}_{-j})] \mid b_j \in [z-\delta, z]] \\
&= \lim_{\xi\to z^-} \mathop{\mathbb{E}}_{\mathbf{b}_{-j}\sim\mathbf{B}_{-j}}[u_j(\xi, \mathbf{b}_{-j})] \\
&= \lim_{\xi\to z^-} \mathop{\mathbb{E}}_{\mathbf{b}_{-j}\sim\mathbf{B}_{-j}}[x_j(\xi, \mathbf{b}_{-j})] \lim_{\xi\to z}(v_j - \xi) \\
&= d_j \lim_{\xi\to z^-} \hat{F}_j^{avg}(\xi)(v_j - z) \\
&< d_j \lim_{\xi\to z^+} \hat{F}_j^{avg}(\xi)(v_j - z)
\end{aligned}
$$

The last equality in the above expressions is by Fact 7. The last inequality holds because $\hat{F}_j^{avg}$ has a discontinuity at z due to the fact that i assigns positive probability to z.

To conclude, the above series of equations imply that there exists a small enough ϵ such that

$$
\mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}}[u_j(b_j, \mathbf{b}_{-j}) \mid b_j \in [z-\delta, z]] < \mathop{\mathbb{E}}_{\mathbf{b}_{-j}\sim\mathbf{B}_{-j}}[u_j(z+\epsilon, \mathbf{b}_{-j})]
$$

which contradicts $\mathbf{B}$ being an equilibrium.

Regarding the second part of the statement, by Observation 12 it cannot be that two bidders have a mass point on $\ell(\mathbf{B}_W)$. For the sake of contradiction, suppose that there exists a bidder i with $Pr[b_i = \ell(\mathbf{B}_W)] > 0$ and i is not a non-empty-handed bidder. Then, the rest of the bidders in $W(\mathbf{B})$ bid higher than $\ell(\mathbf{B}_W)$ with probability one and therefore, bidder i doesn't win any unit by bidding $\ell(\mathbf{B}_W)$. By Theorem 13, $\mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})] = \mathbb{E}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(\ell(\mathbf{B}_W), \mathbf{b}_{-i})] = 0$, which contradicts the fact that $i \in W(\mathbf{B})$. $\square$

Proof of Corollary 23. Suppose for contradiction that there exists an interval $(\ell', h') \nsubseteq \bigcup_{j\in W(\mathbf{B})\setminus\{i\}} Supp(B_j)$ with $\ell' > \ell(\mathbf{B}_W)$, $h' < h(\mathbf{B}_W)$ and this is maximal. Then, let i be a bidder with h' in their support. By Theorem 22, $\hat{F}_i^{avg}(h') = \frac{u_i}{d_i(v_i - h')}$ and $\mathbb{E}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})] = \mathbb{E}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(h', \mathbf{b}_{-i})] = u_i$ by Corollary 20.

For any $x \in (\ell', h')$, $\hat{F}_i^{avg}(x) = \frac{u_i}{d_i(v_i - h')}$, since $(\ell', h') \not\subseteq \bigcup_{j \in W(\mathbf{B}) \setminus \{i\}} Supp(B_j)$. Clearly, $\hat{F}_i^{avg}(x) > \frac{u_i}{d_i(v_i - x)}$ and by bidding x,

$$\mathop{\mathbb{E}}_{\mathbf{b}_{-i} \sim \mathbf{B}_{-i}} [u_i(x, \mathbf{b}_{-i})] = d_i \hat{F}_i^{avg}(x)(v_i - x) > d_i \frac{u_i}{d_i(v_i - x)}(v_i - x) = u_i = \mathop{\mathbb{E}}_{\mathbf{b} \sim \mathbf{B}} [u_i(\mathbf{b})] \,,$$

which is a contradiction to $\mathbf{B}$ being an equilibrium. □

D Missing proofs from Section 4.1

Proof of Lemma 26. To prove the first statement, by Lemma 18, we have that $Supp(B_1) = Supp(B_2)$. Also, by Corollary 23, we have that $Supp(B_1)$ is an interval. To prove that $\ell(B_1) = 0$, we utilize Lemma 10 which states that $|W(\mathbf{B})| \geq 2$. Thus, with exactly two bidders, we have that $|W(\mathbf{B})| = 2$, and both bidders have positive expected utility. Let $\ell = \ell(B_1) = \ell(B_2)$. If $\ell > 0$, we will argue that one of the players has an incentive to deviate to a lower bid.

Note first that by Observation 12, it cannot be that ℓ is a mass point for both bidders. WLOG suppose that bidder 1 has a mass point at ℓ, which by Fact 11, implies she has a positive expected utility, when playing ℓ. At the same time, bidder 2 bids higher than ℓ with probability equal to 1. Hence, there must be some left over units that bidder 1 wins when bidding ℓ in order to have a positive expected utility. But now this means that bidder 1 would have a profitable transfer of probability mass to 0 in order to have a zero payment while obtaining the same number of units. If neither bidder has mass on ℓ, we can use Corollary 20 to have that the expected utility of bidder 1 at ℓ equals $E[u_1(\mathbf{b})] > 0$. Hence, she wins some units with positive probability when bidding ℓ. But then bidder 1 would win the same number of units by bidding 0 resulting in higher utility, a contradiction to $\mathbf{B}$ being an equilibrium.

To prove the second statement of the lemma, we use Theorem 14 that states that at least one of the bidders, say bidder i, is non-empty-handed and by Lemma 24 we obtain

$$h(B_i) = v_i \frac{d_1 + d_2 - k}{d_i}.$$

□

Proof of Lemma 29. Given i, as specified by the statement of the lemma, the definition of the average distribution of winning bids for the other bidder (denoted as bidder $-i$), and Equation (4) yield

$$\hat{F}_{-i}^{avg}(z) = \frac{\sum_{m=1}^{d_{-i}} \hat{F}_{-i,m}(z)}{d_{-i}} = \frac{k - d_i + (d_1 + d_2 - k)F_i(z)}{d_{-i}} = \frac{v_{-i} - h_i(B_i)}{v_{-i} - z} \tag{15}$$

for all $z \in [\ell(B_i), h(B_i)]$. Therefore, we can have the following series of implications for the expected welfare.

$$
\begin{aligned}
\mathop{\mathbb{E}}_{\mathbf{b}}[x_1(\mathbf{b})v_1 + x_2(\mathbf{b})v_2] &= \mathop{\mathbb{E}}_{\mathbf{b}}[x_{-i}(\mathbf{b})(v_{-i} - v_i) + kv_i] \\
&= (v_{-i} - v_i) \mathop{\mathbb{E}}_{b_{-i}}\left[\mathop{\mathbb{E}}_{b_i}[x_{-i}(b_i, b_{-i})]\right] + kv_i \\
&= d_{-i}(v_{-i} - v_i) \mathop{\mathbb{E}}_{b_{-i}}\left[\hat{F}^{avg}_{-i}(b_{-i})\right] + kv_i \\
&= d_{-i}(v_{-i} - v_i) \mathop{\mathbb{E}}_{b_{-i}}\left[\frac{v_{-i} - h_i(B_i)}{v_{-i} - b_{-i}}\right] + kv_i \\
&= d_{-i}(v_{-i} - v_i) \int_0^{h(B_i)} f_{-i}(z) \frac{v_{-i} - h_i(B_i)}{v_{-i} - z} dz + kv_i \\
&= d_{-i}(v_{-i} - v_i) \left(1 - \int_0^{h(B_i)} F_{-i}(z) \frac{v_{-i} - h(B_i)}{(v_{-i} - z)^2} dz\right) + kv_i.
\end{aligned}
$$

The first equality is by observing that $x_i(\mathbf{b}) = k - x_{-i}(\mathbf{b})$. The third equality follows by Fact 7 whereas the fourth equality is due to Equation (15). The expectation over b_{-i} is then replaced by the integral, since only bidder i could have a mass point. The remaining implications are due to integration by parts, and the proof can be completed by substituting in the last equation the CDF F_{-i}, given by (4). ☐

Full Proof of Theorem 25. The properties established so far imply a full characterization of instances that have inefficient equilibria. To establish Theorem 25, we will group instances into three appropriate classes and we will solve an appropriately defined optimization problem that approximates the Price of Anarchy for each subclass to arbitrary precision.

WLOG, suppose we are given a value profile $(\mathbf{v}, \mathbf{d}) = ((v_1, d_1), (v_2, d_2))$ of k units such that $d_1 \geq d_2$. We define the following two quantities, which we refer to as the *normalized* demands.

$$
\bar{d}_1 = \frac{d_1}{k} > 0 \qquad \bar{d}_2 = \frac{d_2}{k} > 0 \tag{16}
$$

Essentially, we intend to use $v_1, v_2, \bar{d}_1$ and $\bar{d}_2$ as the variables of the optimization problem mentioned before.

Let $\mathbf{B}$ be any inefficient mixed Nash equilibrium. With a slight abuse of notation we view the term $h(B_i)$ as a function of the valuation profile parameters, as established by Lemma 26, and define the functions $h_i(\mathbf{v}, \bar{\mathbf{d}}) = v_i \frac{\bar{d}_1 + \bar{d}_2 - 1}{\bar{d}_i}$ for $i = 1, 2$. We pair these functions with two additional expressions $SW_i(\mathbf{v}, \mathbf{d})$ for $i = 1, 2$ which are (scaled) restatements of the social welfare of an equilibrium (as stated in Lemma 29), solely in terms of the value profile $(\mathbf{v}, \mathbf{d})$ and k, and without dependencies on the underlying equilibrium distributions. The reason we are able to do so, is Theorem 27, which tells us what the CDFs are, in terms of the valuation profile. The exact form of $SW_i(v_1, v_2, \bar{d}_1, \bar{d}_2)$ for $i = 1, 2$ is

$$
SW_i(\mathbf{v}, \bar{\mathbf{d}}) = \bar{d}_{-i}(v_{-i} - v_i) \left(1 - \int_0^{h_i(\mathbf{v}, \bar{\mathbf{d}})} \frac{1}{\bar{d}_1 + \bar{d}_2 - 1} \left(\frac{\bar{d}_i(v_i - h_i(\mathbf{v}, \bar{\mathbf{d}}))}{v_i - z} - (1 - \bar{d}_{-i})\right) \frac{v_{-i} - h_i(\mathbf{v}, \bar{\mathbf{d}})}{(v_{-i} - z)^2} dz\right) + v_i.
$$

To continue, let i be a non-empty handed bidder with a mass point at 0 or if no such bidder exists, let i be any non-empty-handed bidder. Then,

$$
kSW_i(\mathbf{v}, \bar{\mathbf{d}}) = SW(\mathbf{B}). \tag{17}
$$

To proceed, we will distinguish the following two cases.

1. If $\mathbf{B} = (B_1, B_2)$ is such that $Supp(B_1) = Supp(B_2) = [0, v_1 \frac{d_1+d_2-k}{d_1}]$, then, by Equation (17), $SW(\mathbf{B}) = kSW_1(\mathbf{v}, \bar{\mathbf{d}})$ and by the second part of Theorem 27 it must be that $\frac{v_2}{v_1} \geq \frac{d_2}{d_1}$ or, equivalently, in terms of normalized demands as $\frac{v_2}{v_1} \geq \frac{\bar{d}_2}{\bar{d}_1}$. We split the analysis into the following sub-cases:

   (a) When $v_1 > v_2$, the optimal social welfare is determined by allocating bidder 1 her entire demand and, subsequently allocating bidder 2 the leftover units. Therefore, in this case $OPT(\mathbf{v}, \mathbf{d}) = v_1 d_1 + (k - d_1) v_2 = k(v_1 \bar{d}_1 + (1 - \bar{d}_1) v_2)$ and $\frac{OPT(B)}{SW(B)} = \frac{v_1 \bar{d}_1 + (1-\bar{d}_1) v_2}{SW_1(v_1, v_2, \bar{d}_1, \bar{d}_2)}$. Hence, the Price of Anarchy of mixed Nash equilibria for this subclass is upper bounded by the optimal solution to the following problem

$$\begin{aligned} \max_{v_1, v_2, \bar{d}_1, \bar{d}_2} \quad & \frac{v_1 \bar{d}_1 + (1 - \bar{d}_1) v_2}{SW_1(v_1, v_2, \bar{d}_1, \bar{d}_2)} \\ \text{subject to} \quad & 1 > \frac{v_2}{v_1} \geq \frac{\bar{d}_2}{\bar{d}_1} \,. \end{aligned} \tag{18}$$

   (b) Similarly, when $v_1 < v_2$ the optimal social welfare is determined by allocating bidder 2 her entire demand and, subsequently allocating bidder 1 the leftover units. Therefore, in this case $OPT(B) = v_1(k - d_2) + d_2 v_2 = k(v_1(1 - \bar{d}_2) + v_2 \bar{d}_2)$ and the Price of Anarchy for this subclass is upper bounded by the optimal solution to the following problem

$$\begin{aligned} \max_{v_1, v_2, \bar{d}_1, \bar{d}_2} \quad & \frac{v_1(1 - \bar{d}_2) + v_2 \bar{d}_2}{SW_1(v_1, v_2, \bar{d}_1, \bar{d}_2)} \\ \text{subject to} \quad & v_2 > v_1 . \\ & \bar{d}_1 \geq \bar{d}_2 \,. \\ & 1 > \bar{d}_2 \,. \end{aligned} \tag{19}$$

   Note that in this sub-case, we enforce the last constraint that $\bar{d}_2 < 1$ (implicitly enforced in the first sub-case). Since we assumed $d_1 \geq d_2$, there can be no mixed Nash equilibrium with $\bar{d}_2 = 1$, because then both bidders are additive, violating the condition of Theorem 14.

2. The final case we need to consider is equilibria $\mathbf{B} = (B_1, B_2)$ such that $Supp(B_1) = Supp(B_2) = [0, v_2 \frac{d_1+d_2-k}{d_2}]$ when $d_1 < k$ (recall if $d_1 = k$ bidder 1 is non-empty-handed and the support will be as in the first case). As in the previous case, by the second part of Theorem 27, it must be that $\frac{v_1}{v_2} \geq \frac{d_1}{d_2}$. However, unlike the class of equilibria described in the previous paragraph, it is sufficient to consider here only the case when $v_1 > v_2$ since, due to our assumption that $d_1 \geq d_2$, the condition $\frac{v_1}{v_2} \geq \frac{d_1}{d_2}$ implies that there cannot exist mixed Nash equilibria when $v_1 < v_2$. Thus, the Price of Anarchy for this subclass is upper bounded by

$$\begin{aligned} \max_{v_1, v_2, \bar{d}_1, \bar{d}_2} \quad & \frac{v_1 \bar{d}_1 + (1 - \bar{d}_1) v_2}{SW_2(v_1, v_2, \bar{d}_1, \bar{d}_2)} \\ \text{subject to} \quad & \frac{v_1}{v_2} \geq \frac{\bar{d}_1}{\bar{d}_2} \geq 1 \\ & 1 > \bar{d}_2 \,. \end{aligned} \tag{20}$$

By solving numerically the optimization problems of Equations (18), (19) and (20), we found out that the worst case instances arise by the sub-case given by (18). In particular, the maximum value for the objective function we obtained was approximately 1.1095 and the optimal values for the four variables are $v_1 = 1, v_2 \approx 0.526, \bar{d}_1 = 1, \bar{d}_2 \approx 0.357$. This concludes the proof of the upper bound on the Price of Anarchy. Furthermore, it is not hard to convert the variables to the underlying worst case instance, which we present in the next paragraph.

Tight Example. Consider an instance of the discriminatory auction for $k \geq 2$ units and $n = 2$ bidders. Bidder 1 has value $v_1 = 1$ and $d_1 = k$, whereas bidder 2 has a value $v_2 = 0.526$ and $d_2 = \lceil 0.357k \rceil$ units. Let B_1, B_2 be two distributions supported in $[0, \frac{d_2}{k}]$. Note that $v_2 > \frac{d_2}{k}$. In accordance to Equation (4), the cumulative distribution functions of B_1 and B_2 are

$$F_1(z) = \frac{v_2 - \frac{d_2}{k}}{v_2 - z} \qquad\qquad F_2(z) = \frac{k - d_2}{d_2} \frac{z}{1 - z}.$$

It is easy to verify that $\mathbf{B} = (B_1, B_2)$ is indeed a mixed equilibrium. The optimal allocation is for bidder 1 to obtain all k units and the expected social welfare of $\mathbf{B}$ can be easily derived using Lemma 29. The worst case inefficiency ratio occurs as k grows and is approximately 1.1095. Therefore, the Price of Anarchy bound described by the optimization problem of Equation (18) is tight and the proof is concluded. $\square$

E Missing proofs from Section 4.2

Proof of Theorem 30. Consider a discriminatory auction instance of $k \geq 2$ units. Let the number of bidders be $n + 1$: one additive bidder, denoted by α, that competes against $n < k$ unit-demand bidders. We assume that ties are in favor of bidder α. The value per unit of the additive bidder is 1, whereas the value of the i-th unit-demand bidder is v_i, for $i = 1, \ldots, n$. The values of the unit-demand bidders are sorted in increasing order, i.e. $v_1 \leq v_2 \leq \cdots \leq v_n$ and $v_n \leq 1$. For convenience, we define, for $i = 0, \ldots, n$, the auxiliary terms $h_i = \frac{i}{k-n+i}$. Moreover, we will later choose the values so that they satisfy the following set of inequalities:

1. For $i = 1, \ldots, n-1,\ m = i+1, \ldots, n$, and every $z \in [h_{m-1}, h_m]$:

$$\prod_{j=i+1}^{m-1} \frac{v_j - h_j}{v_j - h_{j-1}} \geq \frac{v_i - z}{v_i - h_i} \frac{v_m - h_{m-1}}{v_m - z}. \tag{21}$$

2. For $i = 2, \ldots, n,\ m = 1, \ldots, i-1$, and every $z \in [h_{m-1}, h_m]$:

$$\prod_{j=m+1}^{i-1} \frac{v_j - h_j}{v_j - h_{j-1}} \leq \frac{v_m - z}{v_m - h_m} \frac{v_i - h_{i-1}}{v_i - z}. \tag{22}$$

Let $\mathbf{B}$ be a product distribution. The additive bidder α bids according to a distribution B_α supported in $[0, h_n]$ with the cumulative distribution function F_α. F_α is a branch function with n branches, where for $i = 1, \ldots, n$, the form of F_α at $[h_{i-1}, h_i]$, denoted by F_α^i, is

$$F_\alpha(z) = F_\alpha^i(z) = \prod_{j=i+1}^{n} \left(\frac{v_j - h_j}{v_j - h_{j-1}} \right) \frac{v_i - h_i}{v_i - z}.$$

The distribution B_i of each unit-demand bidder $i = 1, \ldots, n$, is supported in $[h_{i-1}, h_i]$, and the form of its CDF is

$$F_i(z) = \frac{k-n}{1-z} - (k - n + i - 1)$$

We now show that this construction is indeed a mixed Nash equilibrium, with the following lemma.

Lemma 49. The profile $\mathbf{B}$ is an equilibrium, provided the values $v_1, \ldots, v_n$ satisfy Equations (21) and (22).

Proof. Firstly, when the additive bidder bids the rightmost point in her support $h_n = \frac{n}{k}$, this grants her an allocation of k (since she outbids all the unit-demand bidders) and an expected utility of $k(1 - \frac{n}{k}) = k - n$. Therefore, bidding above h_n is a dominated strategy for bidder α, since she would still win k units but will be asked to pay more than $h_n = \frac{n}{k}$. On the other hand for $i = 1, \ldots, n$, bidding $z \in [h_{i-1}, h_i)$, grants bidder α an expected utility of

$$\mathop{\mathbb{E}}_{\mathbf{b}_{-\alpha} \sim \mathbf{B}_{-\alpha}} [u_\alpha(z, \mathbf{b}_{-a}) | z \in [h_i - 1, h_i)] = (k - n + i - 1 + F_i(z)) (1 - z) = k - n.$$

Therefore, by taking the expectation over B_α on both sides of this equation, we obtain that $\mathbb{E}_{\mathbf{b} \sim \mathbf{B}} [u_\alpha(\mathbf{b})] = k - n$ and bidder α has no profitable unilateral deviation.

We now examine the incentives for unilateral deviations of the n unit-demand bidders. For each one of the unit-demand bidders $i = 1, \ldots, n$, their expected utility for bidding in the interval of their support (h_{i-1}, h_i) is

$$\mathop{\mathbb{E}}_{\mathbf{b}_{-i} \sim \mathbf{B}_{-i}} [u_i(z, \mathbf{b}_{-i}) | z \in (h_{i-1}, h_i)] = F_\alpha^i(z)(v_i - z) = \prod_{j=i+1}^{n} \left(\frac{v_j - h_j}{v_j - h_{j-1}} \right) (v_i - h_i)$$

and, by taking an expectation on both sides of the above equation, the expected utility of unit-demand bidder i is

$$\mathop{\mathbb{E}}_{\mathbf{b} \sim \mathbf{B}} [u_i(\mathbf{b})] = \prod_{j=i+1}^{n} \left(\frac{v_j - h_j}{v_j - h_{j-1}} \right) (v_i - h_i) \,.$$

Similarly to the additive bidder, no unit-demand bidder is willing to bid higher than h_n, since, even though this strategy will result in outbidding all other bidders and thus guaranteeing them their unit, it will result in overpaying. Moreover, bidding 0 would result in losing to the additive bidder α since ties are in favor of the additive bidder. Finally, no bidder would ever bid v_i or above since such a deviation would result in a non-positive expected utility.

To conclude the proof that this construction is a mixed Nash equilibrium, we need to examine whether any unit-demand bidder i has an incentive to bid outside her support without exceeding h_n. For $i = 1, \ldots, n-1$, suppose that the unit-demand bidder i is unilaterally deviating to a point $z \in [h_{m-1}, h_m]$ such that $z < v_i$, and $m \in \{i+1, \ldots, n\}$. But then, since the value vector is such that Equation (21) holds, we have that

$$\begin{aligned}
\mathop{\mathbb{E}}_{\mathbf{b}_{-i} \sim \mathbf{B}_{-i}} [u_i(z, \mathbf{b}_{-i})] = F_\alpha^m(z)(v_i - z) &= \prod_{j=m+1}^{n} \left(\frac{v_j - h_j}{v_j - h_{j-1}} \right) \frac{v_m - h_m}{v_m - z} (v_i - z) \\
&\leq \prod_{j=i+1}^{n} \left(\frac{v_j - h_j}{v_j - h_{j-1}} \right) (v_i - h_i) = \mathop{\mathbb{E}}_{\mathbf{b} \sim \mathbf{B}} [u_i(\mathbf{b})] \,.
\end{aligned}$$

Finally, for $i = 2, \ldots, n$, consider the unilateral deviation of bidder i to an interval $[h_{m-1}, h_m]$ for $m \in \{1, \ldots, i-1\}$. However, due to Equation (22) we once again obtain that

$$\mathop{\mathbb{E}}_{\mathbf{b}_{-i} \sim \mathbf{B}_{-i}} [u_i(z, \mathbf{b}_{-i})] \leq \mathop{\mathbb{E}}_{\mathbf{b} \sim \mathbf{B}} [u_i(\mathbf{b})] \,.$$

Hence, no unit-demand bidder has a unilateral deviation and $\mathbf{B}$ is a mixed Nash Equilibrium. $\square$

Note that the welfare maximizing allocation is to assign all units to bidder α. Therefore the optimal social welfare is k. To obtain the worst case instance, given a number of units $k \geq 2$, and a number of unit-demand bidders $n < k$, we need to specify a value vector $\mathbf{v}$ so that the expected social welfare is minimized and Equations (21) and (22) hold. By Lemma 41, we can easily obtain the expression of the expected social welfare (normalized by k) and, therefore, the optimization problem that yields the most inefficient auction instance attainable with the above construction given a number of units $k \geq 2$ and a number of unit-demand bidders $n < k$ is

$$
\begin{aligned}
\min_{\mathbf{v}\in(\mathbf{0},\mathbf{1})} \quad & 1-(1-\frac{n}{k})\sum_{i=1}^{n}\prod_{j=i+1}^{n}\left(\frac{v_j-h_j(n,k)}{v_j-h_{j-1}(n,k)}\right)\int_{h_{i-1}(n,k)}^{h_i(n,k)}\frac{v_i-h_i(n,k)}{v_i-z}\frac{1-v_i}{(1-z)^2}dz \\
\text{subject to} \quad & h_i(n,k)=\frac{i}{k-n+i}, \qquad \forall i\in\{0,\dots,n\} \\
& v_i > h_i(n,k), \qquad \forall i\in\{0,\dots,n\} \\
& \prod_{j=i+1}^{m-1}\frac{v_j-h_j}{v_j-h_{j-1}} \geq \frac{v_i-z}{v_i-h_i}\frac{v_m-h_{m-1}}{v_m-z}, \qquad \forall i\in\{1,\dots,n-1\}, m\in\{i+1,\dots,n\}, \\
& \qquad\qquad z\in[h_{m-1}(n,k),h_m(n,k)] \\
& \prod_{j=m+1}^{i-1}\frac{v_j-h_j}{v_j-h_{j-1}} \leq \frac{v_m-z}{v_m-h_m}\frac{v_i-h_{i-1}}{v_i-z}, \qquad \forall i\in\{2,\dots,n\}, m\in\{1,\dots,i-1\}, \\
& \qquad\qquad z\in[h_{m-1}(n,k),h_m(n,k)]
\end{aligned}
\tag{23}
$$

We were able to numerically solve a series of optimization problems of the above format given integers k, n using global optimization routines of the scientific computing library Scipy of Python. We observed that the worst case instances were those in which the ratio $\frac{n}{k} \approx 37\%$. For instance, when $k = 10$ and $n = 4$, the above optimization problem yields 0.8941 and therefore the worst case ratio is $1/0.8941 \approx 1.1184$, which is already higher than the Price of Anarchy attainable with two bidders that we have discussed in Section 4.1. If we increase the number of units further to, say, $k = 100$ and set $n = 37$, the optimization problem yields approximately 0.8925 and therefore the worst case inefficiency becomes $1/0.8925 \approx 1.1204$. By experimenting with very large values of k and n, the worst case inefficiency differs only in the 5th decimal digit, hence we have a convergence to 1.1204, using $\frac{n}{k} \approx 0.3732$.

□

Proof of Lemma 35. For the first statement, note that it cannot be the case that the set difference between $[\ell(\mathbf{B}_W), h(\mathbf{B}_W)]$ and $Supp(B_\alpha)$ is a collection of isolated points, since the distributions utilize closed intervals. Suppose now that there exists an interval $I \notin Supp(B_\alpha)$, with $I \subseteq [\ell(\mathbf{B}_W), h(\mathbf{B}_W)]$. We can choose I to be sufficiently small, so that there exists a bidder $i \in W(\mathbf{B}) \setminus \{\alpha\}$ such that $I \subseteq Supp(B_i)$. This is feasible, since by Corollary 23 the union of the supports of bidders in $W(\mathbf{B}) \setminus \{\alpha\}$ is an interval. Assuming $I = [\ell', h']$, where we can also enforce that $\ell' > \ell(\mathbf{B}_W)$, we obtain

$$
\begin{aligned}
\mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})] &= \mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(h',\mathbf{b}_{-i})] = (v_i-h')d_i\hat{F}_i^{avg}(h') = (v_i-h')d_i\hat{F}_i^{avg}(\ell') \\
&< \mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}}[u_i(\ell',\mathbf{b}_{-i})] = \mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}}[u_i(\mathbf{b})].
\end{aligned}
$$

The first and last equalities above are due to Corollary 20 (since $\ell' > \ell(\mathbf{B}_W)$), the second equality holds by Fact 7, and the third equality follows by Observation 34 and the fact that $F_\alpha(h') = F_\alpha(\ell')$ (because

$I \notin Supp(B_\alpha)$) and therefore $\hat{F}_i^{avg}(h') = \hat{F}_i^{avg}(\ell')$. The contradiction we get establishes the first statement of the theorem.

To prove the second statement, suppose for contradiction that there exist two bidders $i, j \in W(\mathbf{B}) \setminus \{\alpha\}$ such that $v_i \neq v_j$ and an interval $I \subseteq Supp(B_i) \cap Supp(B_j)$. By Theorem 22, we obtain $\hat{F}_i^{avg}(z) = \frac{u_i}{di(v_i - z)}$ and since again by Observation 34, $\hat{F}_i^{avg}(z) = F_\alpha(z)$, we conclude that $F_\alpha(z) = \frac{u_i}{di(v_i - z)}$ for $z \in I$.

Now for bidder j, and every $z \in I$ we obtain

$$\mathop{\mathbb{E}}_{\mathbf{b}_{-j} \sim \mathbf{B}_{-j}}[u_j(z, \mathbf{b}_{-i})] = d_j F_\alpha(z)(v_j - z) = d_j \frac{u_i}{di(v_i - z)}(v_j - z).$$

The right hand side must be the same for all $z \in I$ by Corollary 20. However, this is a contradiction since this can only be true for an infinite set of values for z, only when $v_i = v_j$. $\square$

Proof of Theorem 37. We first construct the value profile $(\mathbf{v}', \mathbf{d}')$ and the product distribution $\mathbf{B}'$. We then argue that they follow the three properties in the statement of the theorem. Firstly, we construct the valuation profile $(\mathbf{v}', \mathbf{d}')$ by modifying the profile $(\mathbf{v}, \mathbf{d})$ as follows: we *replace* every bidder $i \in W(\mathbf{B}) \setminus \{\alpha\}$ with d_i unit-demand bidders, each of them having value v_i. All other bidders (the additive bidder and the losing bidders) retain their values and demands.

We construct the product distribution $\mathbf{B}'$ from $\mathbf{B}$ as follows. We let bidders α and $\mathcal{N} \setminus W(\mathbf{B})$ to bid as before. This leaves us only the newly generated unit-demand bidders. These bidders use the same CDF, and on the same support, as the bidder they were derived from. This completes the description of $\mathbf{B}'$.

Now that we have defined $\mathbf{B}'$ the first property follows easily by observing that the bidders with positive expected utility are precisely all the newly generated unit-demand bidders as well as bidder α.

To see that $\mathbf{B}'$ is an equilibrium, note that the losing bidders have no incentives to deviate, just as in $\mathbf{B}$. Since the CDF of bidder α has not changed, the unit-demand bidders have no incentive to deviate because they face the same competition from α as the bidders in $\mathbf{B}$. If there was a successful deviation by a unit-demand bidder, this would directly translate to a deviation in $\mathbf{B}$. The same is true for the additive bidder since she also sees the same competition on average, and this does not affect the improvement of her expected utility .

Finally, it is very easy to see that $SW(\mathbf{B}) = SW(\mathbf{B}')$, and also that $OPT(\mathbf{v}, \mathbf{d}) = OPT(\mathbf{v}', \mathbf{d}')$, which establishes the last statement. $\square$

Proof of Lemma 38. If $\mathbf{B}$ is such that no support intervals intersect for unit-demand bidders, we are done. Otherwise, there exists an interval in which more than one unit-demand bidders bid. Let $S \subseteq W_{\mathbf{B}} \setminus \{\alpha\}$ be such a set of unit-demand bidders and let $I_j = \bigcup_{i \in S} Supp(B_i)$.

Consider the perspective of bidder α when she bids at $I_j \in [L, R]$. Her average CDF of winning bids when bidding in I_j is by Theorem 22

$$\hat{F}_\alpha^{avg}(z) = \frac{D_j(v_{\alpha - R})}{k(v_\alpha - z)}.$$

Suppose that we partition the interval $[L, R]$ to $D_j - Dj - 1$ disjoint subintervals and assign each bidder to one of them. For $i = 1, \ldots, D_j - D_{j-1}$, the CDF of bidder i will be such that the equation above remains satisfied. This means that

$$F_i(z) = \frac{(D_j)(v_a - R)}{v_\alpha - z} - (k - D_{j-1} - (i - 1)),$$

and for every two $p_i, pi + 1$ it must be that

$$(k - n + D_{j-1} + i)(v_\alpha - p_i) = (k - n + D_{j-1} + i + 1)(v_\alpha - p_{i+1})$$

These points are clearly inside the interval $[L, R]$.

We partition the interval I_j into $|S|$ intervals. Then, we assign a unit demand bidder in S to bid in a different subinterval with the CDF function H_s. Call this new product distribution B'. Similarly to Theorem 37, it is clear that the incentives of the small bidders remain unchanged since bidder α did not change their distribution. The same is true for bidder α due to the transformation we defined above. This concludes the proof. $\square$

Proof of Lemma 41. For brevity we denote $\ell(B_a)$ as ℓ and, for $j = 1, \ldots, n$, we denote $h(B_j)$ as h_j.

We have that

$$\begin{aligned} SW(\mathbf{B}) &= \mathop{\mathbb{E}}_{b_\alpha \sim B_\alpha}\left[\mathop{\mathbb{E}}_{\mathbf{b}_{-\alpha} \sim \mathbf{B}_{-\alpha}}\left[\sum_{i=1}^{n} x_i(\mathbf{b}) v_i\right]\right] \\ &= F_\alpha(\ell)(k - n + \sum_{j=1}^{n} v_j) + \sum_{i=1}^{n} \int_{h_{i-1}}^{h_i} f_{\alpha^i(z)} \left(F_i(z)(v_\alpha - v_i) + v_\alpha(k - n + i - 1) + \sum_{j=i}^{n} v_j \right) dz \\ &= F_\alpha(\ell)(k - n + \sum_{j=1}^{n} v_j) + \sum_{i=1}^{n} \int_{h_{i-1}}^{h_i} f_{\alpha^i(z)} \left(F_i(z)(v_\alpha - v_i) + v_\alpha(k - n + i - 1) + \sum_{j=i}^{n} v_j \right) dz \\ &= F_\alpha(\ell)(k - n + \sum_{j=1}^{n} v_j) + \sum_{i=1}^{n} \int_{h_{i-1}}^{h_i} f_{\alpha^i(z)} \left(\frac{k(v_\alpha - h_n)}{v_a - z}(v_\alpha - v_i) + v_i(k - n + i) + \sum_{j=i+1}^{n} v_j \right) dz \end{aligned}$$

By integrating the integral by parts, rearranging terms and substituting F_{α^i} by Equation 8 we obtain Equation 12. $\square$

F Missing proofs from Section 5

Proof of Theorem 43. Consider an instance of the discriminatory auction for $k \geq 2$ units and $n = 2$ bidders. Let $d \in \{1, \ldots, k-1\}$ be an integer to be specified later and let $h = \frac{d}{k} < 1$. The type of bidder 1 is $(1, k)$ (i.e., deterministically additive with a value of 1 for each of the k units). The type of bidder 2 is (v_2, d) where v_2 is drawn from a continuous distribution V_2. Both bidders reveal their demands. Bidder 1 bids a mixed strategy according to the distribution B_1 supported in $[0, h]$. We denote the (continuous) CDF of B_1 as F_1 and will present its formula in the sequel. Moreover, we denote by f_1 its probability density function.

Given a value v_2 drawn from V_2, the second bidder bids according to some bidding function $b_2(v_2)$ that maps the drawn value v_2 to a bid. Therefore, for each value v_2, bidder 2 specifies a pure strategy bid. Nevertheless, due to the randomness of the value distribution V_2, bidder 1 competes against mixed strategies and observes a CDF

$$F_2(x) = \frac{x}{1-x}\frac{1-h}{h}$$

that describes the distribution of the random variable $b_2(v_2) \in [0, h]$ which we denote as $B_2(V_2)$.

Optimality of bidding function $b_2(\cdot)$ Consider the utility maximization problem of bidder 2. For all values v_2 drawn from distribution V_2, bidder 2 must, at a Bayes Nash equilibrium, specify a pure bid $b_2 \in [0, h]$ that maximizes her expected utility, i.e.

$$\max_{b_2 \in [0,h]} \mathop{\mathbb{E}}_{b_1 \sim B_1}[x_2(b_1, b_2)](v_2 - b_2) \Leftrightarrow \max_{b_2 \in [0,h]} dF_1(b_2)(v_2 - b_2),$$

and the equivalence is due to the fact that the sole source of competition for bidder 2 is bidder 1. Working backwards and viewing the value of bidder 2 as a function of her bid (which is the inverse of the bidding function), and taking first order conditions with respect to b_2, we obtain that

$$v_2(b_2) = b_2 + \frac{F_1(b_2)}{f_1(b_2)}, \qquad b_2 \in (0, h]. \tag{24}$$

Hence, a utility-maximizing bidding function satisfies Equation (24).

We now argue that this auction instance is a Bayes Nash Equilibrium. Firstly, neither bidder has an incentive to bid above h, as bidding h already guarantees them their entire demand. Moreover, lying about one's demand is also a weakly dominated strategy for both bidders.

When bidder 1 declares a bid $z \in [0, h]$, her utility is

$$\mathop{\mathbb{E}}_{v_2 \sim V_2}[u_1(z, b_2(v_2))] = (k - d + dF_2(z))(1 - z) = k - d$$

Therefore, since the expected utility of bidder 1 is $k - d$ and is a constant at every subinterval of her support, bidder 1 has no incentive to deviate unilaterally.

In the case of bidder 2, we have chosen her bidding function $b_2(v_2)$ to be one that satisfies Equation (24). Since this bidding function maximizes her utility, it holds that, given a type v_2

$$\mathop{\mathbb{E}}_{b_1 \sim B_1}[x_2(b_1, b_2(v_2)](v_2 - b_2(v_2)) \geq \mathop{\mathbb{E}}_{b_1 \sim B_1}[x_2(b_1, b')](v_2 - b')$$

for all $b' \in [0, h]$. Therefore, this instance is a Bayes Nash Equilibrium.

The expected social welfare of this BNE is

$$\begin{aligned}
\mathop{\mathbb{E}}_{b_2 \sim B_2(V_2)}\left[\mathop{\mathbb{E}}_{b_1 \sim B_1}[x_1(b_1, b_2) + x_2(b_1, b_2)v_2(b_2)]\right] &= \int_0^h f_2(z)\left(\mathop{\mathbb{E}}_{b_1 \sim B_1}[x_1(b_1, z) + x_2(b_1, z)v_2(z)]\right) dz \\
&= \int_0^h f_2(z)\left(k - \mathop{\mathbb{E}}_{b_1 \sim B_1}[x_2(b_1, z)](1 - v_2(z))\right) dz \\
&= k\int_0^h f_2(z)\left(1 - hF_1(z)\left(1 - z - \frac{F_1(z)}{f_1(z)}\right)\right) dz \\
&= \frac{k(1-h)}{h}\int_0^h \frac{1}{(1-z)^2}\left(1 - hF_1(z)\left(1 - z - \frac{F_1(z)}{f_1(z)}\right)\right) dz
\end{aligned} \tag{25}$$

The third equality in the above derivation is due to Equation (24). In Equation (25) the social welfare of this instance is written in terms of $h = \frac{d}{k}$ and the functions F_1 and f_1.

We have already shown that this instance is in fact a Bayes Nash Equilibrium for any continuous function F_1 supported in $[0, h]$. The question remains which function F_1 (and consequently f_1 as its derivative) should we choose. Ideally, we would want to pick the continuous and increasing function F_1 that minimizes the expected social welfare as long as $F_1(h) = 1$. This is possible, as Equation (25) is a well-posed problem of variational calculus, a field of mathematics with a goal of finding functional maxima and minima. Using such an approach (solving the Euler-Lagrange equation of the problem), we were able to determine that the function

$$F_1(x) = \frac{\mathcal{W}\left(-e^{-1}(1-h)^2\right)}{\mathcal{W}(-e^{-1}(1-x)^2)}$$

is such a functional minimum for Equation (25).

Therefore, by replacing F_1 and f_1 into Equation (25) and subsequently optimizing with respect to $h \in (0,1)$ we obtain a value of approximately $0.8925k$ for the expected social welfare. Finally we observe that for all $z \in [0,h]$ it holds that

$$v(z) = z + \frac{F(z)}{f(z)} = z + (1-z)(1 + \mathcal{W}(-e^{-1}(1-z)^2)) \leq 1$$

and therefore, the optimal social welfare is k. Thus we have obtained a bound of 1.1204 and the proof follows. $\square$